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ABSTRACT

This is part one of a three-part teacher's guide for an SMSG text designed to be used as a one-semester course for twelfth-grade students. This guide includes a suggested time schedule, a teacher's commentary, and answers to exercises. (MK)

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ANALYTIC GEOMETRY

Teachers' Commentary

Part 1

(revised edition)

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ANALYTIC GEOMETRY

Part 1

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Contents

INTRODUCTION 1

A WORD ABOUT THE EXERCISES 2

A SUGGESTED TIME SCHEDULE 3

Chapter

1. ANALYTIC GEOMETRY 5
2. COORDINATES AND THE LINE 7
3. VECTORS AND THEIR APPLICATIONS 83
4. PROOFS BY ANALYTIC METHODS 123
5. GRAPHS AND THEIR EQUATIONS 147

INTRODUCTION

The text Analytic Geometry had its beginnings in 1962 when a small committee of mathematicians and teachers met to discuss the question as to whether there was a need for a new text in analytic geometry for high school, and whether the School Mathematics Study Group should undertake to write one. Since the conclusion was affirmative, some guidelines were prepared to indicate the form and content desired.

In the summer of 1963 an experimental text and accompanying commentary were prepared by an SMSG writing team consisting of university mathematicians and high school teachers. During the following school year this text was used by about 30 teachers in schools distributed from California to New England, but mostly in 2 centers where the teachers had the benefit of conferences with each other and with an interested college professor. The complete revision of the text and commentary in the summer of 1964 took into account both the comments and criticisms of these teachers, and the recommendations of an advisory committee of the SMSG Board. We are deeply indebted to those who helped with suggestions, especially to the teachers who used the experimental text.

Analytic Geometry is intended for use as a one-semester course in the 12th grade. It is expected that the students would have completed SMSG Intermediate Mathematics or the equivalent. If it is planned to use Elementary Functions with the same class, it is suggested that that text be used before the Analytic Geometry. However, knowledge of Elementary Functions has not been assumed in this text.

The suggested time schedule here is only tentative; the teacher will adapt it to the particular class. Certain topics are presented here for completeness; for example, some of the work on forms of an equation of a line, on conic sections, or on vectors, will have been studied previously by many classes. Very little time need be spent on familiar work, giving more time for new topics or for supplementary work.

We believe that a reasonably well-prepared class of the students who elect 12th grade mathematics can complete our basic text (Chapters 1 to 10) in a semester. The material in the supplementary chapters was placed there because it was not felt essential to the continuity of the course. However, we feel that this is important and interesting material; we think that it is within the grasp of able students and will broaden their mathematical background.

It is hoped that good classes and individual able students will use the supplementary chapters.

Following the opening remarks for each chapter in this Commentary, you will find running comments keyed in the margin to the pages of the student's text. These contain further explanation and background which we hope will be useful to you.

A WORD ABOUT THE EXERCISES

Some of the exercises are designed to provide just exercise, but you will find that some others are far from routine. Within each set of exercises the arrangement is usually from the more routine to the more complex problems. The most difficult problems are listed separately as "Challenge Problems". A few problems have been included which extend the material beyond the regular textual treatment. We advise you to look at each such problem before assigning it to a student so that you may ascertain whether it is appropriate and how much time it will consume.

We cannot suggest appropriate class assignments since they will vary with the preparation and ability of the class. Of course, enough drill work should be included to fix the fundamental skills and concepts. In the case of a well-prepared class, the drill-type problems might be omitted entirely on any topic previously studied. While the particular problems assigned will vary with the class and perhaps even with the individual pupils, it is hoped that all students will be assigned some of the problems which may be more time consuming but which will show them some of the "fun" of Analytic Geometry.

Solutions for the exercises appear at the point in the running commentary corresponding to the placement of the problems in the student's text. Any given problem may have several acceptable solutions; therefore, the solution presented here should not be considered as the "right", or only, solution. The student is encouraged frequently to use his own judgment in pursuing a solution; hence, if he presents a solution which is correct, it should be accepted.

A SUGGESTED TIME SCHEDULE

The basic text (Chapters 1 to 10) was designed to be covered in one semester of eighteen weeks. The time schedule given below is the result of combining the opinions of the authors with the experience of the teachers who used the preliminary edition.

If you find that your class is falling behind the suggested schedule, you may wish to compensate by treating some topics in less depth or by assigning fewer exercises. If this procedure is not satisfactory, you probably should consider cutting short, first on Chapter 10 and then on Chapter 9. The text was designed so that the least loss to the students would occur in this circumstance.

Chapter	No. of Days	Cummulative Total
1. Analytic Geometry	1	1
2. Coordinates and the Line	10	11
3. Vectors and Their Applications	12	23
4. Proofs by Analytic Methods	8	31
5. Graphs and Their Equations	9	40
6. Curve Sketching and Locus Problems	11	51
7. Conic Sections	9	60
8. The Line and the Plane in 3-space	7	67
9. Quadric Surfaces	10	77
10. Geometric Transformations	8	85

Chapter 1

ANALYTIC GEOMETRY

Chapter 1 is a brief introduction to the text. It is intended to give the students an idea of what analytic geometry is and to show them they already know something about the subject. If possible, they should read it before the first meeting of the class and reread it at intervals during the course.

Since coordinate systems are so important in analytic geometry, it is advisable to discuss in class some of the examples mentioned. The students should be asked to explain latitude and longitude, which are mentioned but not defined in the text. They might be invited to suggest other coordinate systems for a line, a plane, space, a spherical surface, and a torus. However, the coordinate systems which are important in the course are treated in detail later, so not much class time should be spent on them at this point.

Chapter 1 also includes a discussion of the reasons for studying analytic geometry. It is felt that students should know something of the role of analytic geometry among the various branches of mathematics, and that they should realize that their main goal is not information about the particular topics studied, but rather understanding of and ability to use the techniques of analytic geometry.

Analytic Geometry really began when it was realized that every geometric object and every geometric operation can be referred to the number system and, hence, to algebra. The most significant steps in this arithmetization of geometry were taken by two French mathematicians, Pierre Fermat (1601 - 1655) and René Descartes (1596 - 1650). Fermat began work on analytic geometry in 1629 but his treatise Ad Locus Planos et Solidos Isagoge was not published until 1679. Chief credit, therefore, is given to Descartes whose Geometrie was published in 1637 and who influenced the work of many mathematicians. In the Geometrie, one finds the earliest unification of algebra and geometry. Apollonius and other Greek mathematicians had used coordinates to locate points in a geometric figure. It was Descartes who introduced the algebraic representation of a curve or surface by an equation involving two or three variables.

95

Descartes' book does not contain a systematic development of the subject such as you find in this text. The method must be constructed from isolated statements in different parts of the treatise. It is interesting that Fermat's work included the equations $y = mx$, $xy = k$, $x^2 + y^2 = a^2$, $x^2 \pm a^2y^2 = b^2$ for lines and conics.

Many mathematicians extended Descartes' work. Among these were John Wallis in his *Tractatus de Sectionibus Conics* and John DeWitt in his *Elementa Curvarum Linearum*. Most of the work of Descartes and his contemporaries was concerned with the geometry of Apollonius. Newton worked with algebraic equations in his study of cubic curves in 1703. The first analytic geometry of conic sections divorced from the work of Appollonius was developed by Euler in his *Introductio* in 1748.

Since that time the methods of Analytic Geometry have become the most significant in the study of geometry. In more advanced mathematics they have essentially replaced the synthetic method. More recently vector methods have been incorporated in Analytic Geometry and are being used more and more widely in mathematical applications.

Teacher's Commentary

Chapter 2

COORDINATES AND THE LINE

This chapter is fundamental to the rest of the book. In it we discuss coordinate systems for a line and a plane. We also treat the analytic geometry of lines in a plane. A good deal of the material in the chapter is familiar from previous courses; it is repeated here for purposes of review and completeness. You will probably find that the material of Sections 2-1, 2-2, 2-3, and 2-5 may be covered very quickly. It is likely that the material on polar coordinates, direction on a line, angles between lines, and the normal and polar forms of an equation of a line will be new to most students. The majority of the class time should be spent on these topics. Many examples have been interspersed throughout the text. Though these increase the number of pages in the chapter, hopefully they will help the student to proceed more rapidly and decrease the need for classroom explanation and discussion. Many more exercises have been included than any given class might be expected to do. You will probably find it advisable to break the chapter into two units for testing purposes. For this reason a set of review exercises has been included after Section 2-5.

7-15 If the students are to get anything out of this section, they must understand clearly the treatment of distance in SMSG Geometry. By the Distance Postulate, to every pair of different points there corresponds a unique positive number. It is called the distance between the points because it is the "official" version of the intuitive notion of distance. The Ruler and Ruler Placement Postulates enable us to make any point on a line the origin of a coordinate system, and to make either direction from that point the positive one. However, we can not choose the scale. It is already there in the geometry. Betweenness and congruence are defined in terms of coordinates, and thus coordinate systems are fundamental in the development of the SMSG Geometry.

Nevertheless, intuition tells us that scale doesn't really matter. If two boats are equally long, their lengths expressed in meters are equal just as their lengths expressed in feet are equal. Let a, b, and c be the coordinates of the points A, B, and C on a line, in a certain coordinate system, and $a < b < c$. Then if we change the size of the units (but nothing else) in our coordinate system, and a', b', and c' are the new coordinates of the same points, we should find that $a' < b' < c'$. We have not attempted to prove that we do have this freedom in the text. In order to get started on the task before us, we have offered examples illustrating the ways in which we normally assume this freedom in applying geometry. The examples themselves are trivial in difficulty and were deliberately chosen so; their purpose is to illustrate the many assumptions we make in solving even a simple problem as well as the importance of these assumptions.

9 The techniques of analytic geometry are more saleable if we exploit to the fullest the freedom to choose various coordinate systems. When the occasions arise to mention this freedom, we shall make much of it, usually by invoking a grandiose principle as we do here in the Linear Coordinate System Principle.

In this principle we are actually postulating a theorem we could prove, but the proof is difficult for most students. We have included material in the supplement to Chapter 2, for able students who are well versed in SMSG Geometry and the concept of function, and who are interested in the deductive nature of mathematics.

Note that the symbol "d(R,S)" is defined in terms of a fixed coordinate system. It would be nice if our notation showed this, but that would make it rather complicated. It is advisable to stress this point when the symbol is introduced, so the students will be reminded of it every time they see it later.

10 The definition of a directed segment will probably seem rather unnatural to the students. They will feel that the idea of the segment $\overline{AB}$ considered as running from A to B is quite clear and they will wonder why we give this strange definition. It may help to ask them to try to define the concept in terms which are "official" in our formal system. They will find that any definition of this kind, and no other kind is permissible, seems unnatural.

This is not the first time the students have seen such a definition. They undoubtedly felt they knew what the inside of a triangle was before they studied geometry, and most of them were probably surprised to find out how much trouble it was to give an acceptable definition.

Exercises 2-1

1. There should be some agreement between the numbers obtained by comparing these measurements and those numbers in the text. However, the degree of agreement will depend upon how well the subdivisions of the units are estimated. The constants of proportionality should be consistent.

2. The side is measured to 2 place accuracy and the results are correct to 2 place accuracy. The discrepency between 2.53 and 2.54 is not significant because they are the same to 2 place accuracy.

3. Hopefully, students will be able to anticipate that the proper units are feet; the computed answer (12π ft. $= 37.6992$) seems so idealized to be meaningless.

4. The answer will depend upon the source of the information as to the distance from New York to San Francisco. The answer should be close to 400 miles to the inch.

5. 1 inch represents $\approx$ 330 miles; the "line" from New York to San Francisco would be approximately 9.2 inches long.

6. The bicyclist travels at the rate of 8 mi/hour. The friend travels at the rate of 32 km/hour or $\approx$ 20 mi/hour.
   a) $8t - 20(t - 2) =$ distance apart at time t. One hour after the friend begins $(t = 3)$ the distance apart is 4 miles.
   b) When the distances both have traveled are equal, $20(t - 2) = 8t$ and $t = 3\frac{1}{3}$ hours. The distance is (approximately) 27 miles.

7. Rate of bicyclist A is 4 miles/hour.
   Rate of bicyclist B is 5 miles/hour.
   Rate of preposterous bee is 10 miles/hour.
   a)
   $$10t + 5t = 30$$
   $$15t = 30$$
   $$t = 2 \text{ hour}$$
   Distance bee traveled $= 2 \times 10 = 20$ mi.
   b)
   $$4t + 5t = 30$$
   $$9t = 30$$
   $$t = \frac{10}{3} \text{ or } 3\frac{1}{3} \text{ hours}$$
   Total distance bee traveled $= 3\frac{1}{3} \times 10$ or $33\frac{1}{3}$ mi.

16 The statement of the Linear Coordinate System Principle clearly indicates that the measures of distance are proportional, but it is perhaps not so clear that the criteria for order, or betweenness, also carry over in the coordinate systems which we consider. It is not a trivial matter to show that it does. Unfortunately, any numerical example would be hopelessly artificial. An illustration of this idea can be found in physics. The boiling point of alcohol is between the boiling point of water and the freezing point of water. The relationship of betweenness would hold for the corresponding temperatures at these points, whether indicated in the Fahrenheit or the Centigrade scales.

18 The notion of a point of division may be extended to include the endpoints of the segment and points external to the segment, but directed distance should be used in this case in order to assure uniqueness. If in the equation $\frac{\overrightarrow{d}(P,X)}{\overrightarrow{d}(P,Q)} = t$, we define $\overrightarrow{d}(P,X)$ to be the directed distance from P to X and $\overrightarrow{d}(P,Q)$ to be the directed distance from P to Q, we may write

$$\frac{x - p}{q - p} = t .$$

In this case, when $0 < t < 1$, we still obtain internal points of division. When $t = 0$, we obtain the coordinate of P; when $t = 1$, we obtain the coordinate of Q. When $t < 0$, we obtain the coordinates of points in the ray $\overrightarrow{QP}$ which are external to $\overline{PQ}$; when $t > 1$, we obtain points in the ray $\overrightarrow{PQ}$ which are external to $\overline{PQ}$.

18 If your students are like ours, they will comprehend the notion of a weighted average even more clearly when it is applied to test grades which are "weighted" in calculating the final average.

20 There is additional material on linear combinations in the Supplement to Chapter 3 and in SMSG Intermediate Mathematics on pages 374-376 and page 449.

The parametric representation is equivalent to the extension of the notion of point of division given in the note on page 18. If the SMSG Geometry with Coordinates is available, you may wish to look at the material on pages 107-111.

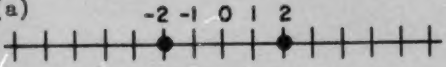
21 The material on the analytic representations of the subsets of a line is more important as an introduction to later work than it is in itself. It provides a review of the notion of the graph of an equation and a reminder

that conditions other than equations also have graphs. If the students are not familiar with the properties of inequalities, it may be necessary to spend a little time on them at this point.

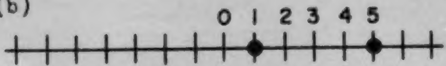
Exercises 2-2

1.

(a)



(b)



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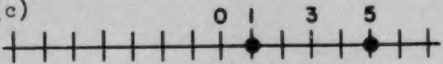

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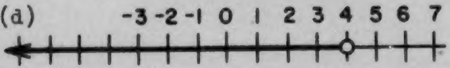
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0
1
2
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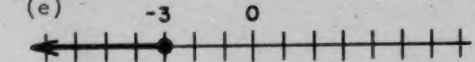
4
5
6
7

(e)



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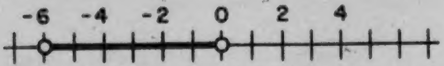

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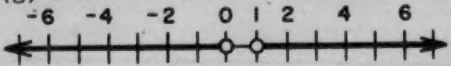

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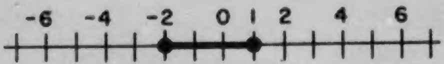

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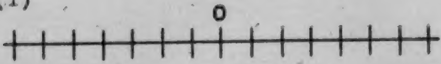

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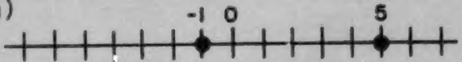

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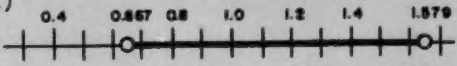
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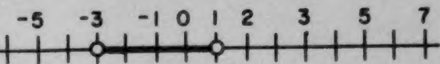
0.8
1.0
1.2
1.4
1.579

(l)



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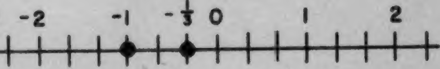
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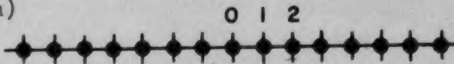
-1
$-\frac{1}{3}$
0
1
2

(n)



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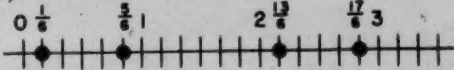
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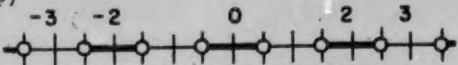
$\frac{1}{6}$
$\frac{5}{6}$
1
2
$\frac{13}{6}$
$\frac{17}{6}$
3

(p)



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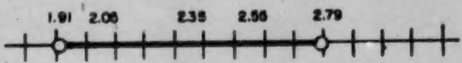
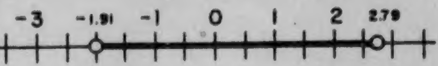

1.91
2.05
2.35
2.55
2.79

(r)


-3
-1.91
-1
0
1
2
2.79

2. (a) $3 \leq x \leq 4$

   Alternative: $(x-3)(x-4) \leq 0$

   (b) $-2 \leq x < 2$

   (c) if $b > a$:

   $x \geq a + 2(b - a) = 2b - a$

   (d) $x \leq x_2 + \frac{1}{2}(x_2 - x_1)$

   (e) $(x+1)(x)(x-1)(x-3)\frac{x}{x} \leq 0$

   Alternative: $-1 \leq x < 0$

   or $1 \leq x \leq 3$

   (f) $(x+2)(x+1)(x-1)(x-3)\frac{(x-1)}{(x-1)} \leq 0$

   (g) $|x - \frac{2}{5}| \leq \frac{1}{5}$

   Alternative: $\frac{1}{5} \leq x \leq \frac{3}{5}$

   (h) $x < 3$

   (i) $\sin \pi \quad x \neq 0$

   (j) $\sin \theta \geq 0$

3. (a) $3a$, $-3a$

   (b) All values of x such that $0 \leq x \leq 1$.

4.

(a)	$m = \frac{15}{2}$	$a = 6$	$b = 9$
(b)	$m = \frac{11}{2}$	$a = 3$	$b = 8$
(c)	$m = r$	$a = r + \frac{1}{3}s$	$b = r - \frac{1}{3}s$
(d)	$m = (r + t) + 1$	$a = (r + t)$	$b = (r + t) + 2$
(e)	$m = r + \frac{3}{2}t$	$a = \frac{4}{3}r + t$	$b = \frac{2}{3}r + 2t$
(f)	$m = \frac{5}{2}r + \frac{1}{2}s$	$a = \frac{7}{3}r + \frac{4}{3}s$	$b = \frac{8}{3}r - \frac{1}{3}s$
(g)	$m = \frac{1}{2}(r^2 - r + s^2 - s)$	$a = \frac{2}{3}(r^2 - r) + \frac{1}{3}(s^2 - s)$	$b = \frac{1}{3}(r^2 - r) + \frac{2}{3}(s^2 - s)$
(h)	$m = \frac{1}{2}(r + s)$	$a = \frac{2}{3}r + \frac{1}{3}s$	$b = \frac{1}{3}r + \frac{2}{3}s$

5. (a) $X = Q$

   (b) $X = P$

   (c) X is between P and Q

   (d) Q is between P and X

   (e) P is between X and Q

   (f) Q is between P and X

6. (a) $t = -1$

$t = \frac{1}{3}$

(b) $t = \frac{2}{3}$

$t = 2$

(c) $t = 3$

$t = -1$

(d) $t = -1$

$t = 1$

7. (a) $$\frac{d(A,B)}{d(B,C)} = \frac{1 - 1\frac{1}{2}}{1\frac{1}{2} - 2\frac{1}{2}} = \frac{-\frac{1}{2}}{-1} = \frac{1}{2}$$

(b) $$\frac{d(B,C)}{d(C,D)} = \frac{1\frac{1}{2} - 2\frac{1}{2}}{2\frac{1}{2} - 4\frac{1}{2}} = \frac{-1}{-2} = \frac{1}{2}$$

(c) $$\frac{d(C,D)}{d(D,E)} = \frac{2\frac{1}{2} - 4\frac{1}{2}}{4\frac{1}{2} - 9} = \frac{-2}{-4\frac{1}{2}} = \frac{4}{9}$$

8. (a) $b = \frac{2}{3}a + \frac{1}{3}c$

(b) $c = \frac{2}{3}b + \frac{1}{3}d$

(c) $d = \frac{9}{13}c + \frac{4}{13}e$

9. (a) $T_1 = 1\frac{1}{2}$ $T_2 = 2$

(b) $T_1 = 2\frac{1}{2}$ $T_2 = 3\frac{1}{2}$

(c) $T_1 = \frac{14}{3}$ $T_2 = \frac{41}{6}$

10. $P = \frac{3}{4}$ or $\frac{9}{4}$

$Q = 1$ or 4

$R = 6$ or 12

26 The teacher will have to use his own judgment as to how much time should be spent on coordinate systems in the plane not of the type we define. For example, if we consider two mutually perpendicular lines and on each of them a perfectly arbitrary linear coordinate system, then by the method described in the text there is established a one-to-one correspondence between the points in the plane and the ordered pairs of real numbers. However, many things become more complicated. The distance between two points, for example, is no longer given by the usual formula. Probably no more than a few minutes should be spent on this in class, after which Challenge Exercise 4 on page 54 can be assigned. (See Supplement C for more on this subject.)

27 We may, of course, extend the notion of point of division as we did on page 18.

29 If the SMSG Geometry with Coordinates is available, you may want to look at pages 543-550, where there is an alternative development of the parametric representation of the points on a line.

Exercises 2-3

1. (a) $M = (3, 4\frac{1}{2})$

   $A = (2,3)$

   $B = (4,6)$

   (b) $M = (5, 7\frac{1}{2})$

   $A = (4,6)$

   $B = (6,9)$

   (c) $M = (5\frac{1}{2}, 2\frac{1}{2})$

   $A = (5\frac{1}{3}, 5\frac{2}{3})$

   $B = (5\frac{2}{3}, -\frac{2}{3})$

   (d) $M = (-2\frac{1}{2}, 3\frac{1}{2})$

   $A = (-\frac{1}{3}, 1\frac{1}{3})$

   $B = (-\frac{14}{3}, \frac{17}{3})$

(e) $M = (0,0)$
$A = (-2,-1)$
$B = (2,1)$

(f) $M = (-4\frac{1}{2}, -4\frac{1}{2})$
$A = (-4,-5)$
$B = (-5,-4)$

(g) $M = \frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}$
$A = \frac{2p_1 + q_1}{3}, \frac{2p_2 + q_2}{3}$
$B = \frac{p_1 + 2q_1}{3}, \frac{p_2 + 2q_2}{3}$

(h) $M = (\frac{3s}{2}, \frac{3t}{2})$
$A = (\frac{5s}{3}, \frac{8t}{3})$
$B = (\frac{4s}{3}, \frac{t}{3})$

(i) $M = (\frac{3r}{2} + \frac{s}{2}, -2r - \frac{s}{2})$
$A = (\frac{7r}{3} + s, -\frac{7r}{3})$
$B = (\frac{2r}{3}, -\frac{5r}{3} - s)$

2. (a) $x = 2a + 6b$
$y = 3a + b$

(b) $x = -4a + 2b$
$y = 5a - 7b$

(c) $x = -3a - 6b$
$y = -6a + 4b$

3. (a) $x = 2 + 4t$
$y = 3 - 2t$

(b) $x = -4 + 6t$
$y = 5 - 12t$

(c) $x = -3 - 3t$
$y = -6 + 10t$

4. If, in equation (2), $x_0 = x_1$ or $y_0 = y_1$

$$x = \frac{dx_0 + cx_0}{c + d}$$

or

$$y = \frac{dy_0 + cy_0}{c + d}$$

Simplifying, $x = x_0$

or $y = y_0$

These are conditions describing points on lines parallel to the y-axis or x-axis respectively.

5. (a) Substituting into equation (1) we see that

$$\frac{7 - (-3)}{22 - (-3)} = \frac{0 - (-6)}{9 - (-6)}$$

$$\frac{10}{25} = \frac{6}{15}$$

$$\frac{2}{5} = \frac{2}{5}$$ $\therefore$ Points A, B, C are collinear

Check:

$$d(A,B) = \sqrt{(7 - (-3))^2 + (0 - (-6))^2} = \sqrt{136} = 2\sqrt{34}$$

$$d(B,C) = \sqrt{((-3) - 22)^2 + ((-6) - 9)^2} = \sqrt{850} = 5\sqrt{34}$$

$$d(A,C) = \sqrt{(7 - 22)^2 + (0 - 9)^2} = \sqrt{306} = 3\sqrt{34}$$

$$d(A,B) + d(A,C) = 2\sqrt{34} + 3\sqrt{34} = d(B,C)$$

$\therefore$ A, B, C must be collinear

(b)

$$\frac{-1 - 3}{-5 - 3} \overset{?}{=} \frac{4 - (-14)}{-6 - (-14)}$$

$$\frac{-4}{-8} \neq \frac{18}{8}$$ not collinear

Check:

$$d(A,B) = \sqrt{((-1) - 3)^2 + (4 - (-14))^2} = \sqrt{340} = 2\sqrt{85}$$

$$d(B,C) = \sqrt{(3 - (-5))^2 + ((-14) - (-6))^2} = \sqrt{128} = 8\sqrt{2}$$

$$d(A,C) = \sqrt{((-1) - (-5))^2 + (4 - (-6))^2} = \sqrt{592} = 4\sqrt{37}$$

$$d(A,B) + d(B,C) \neq d(A,C)$$

This verifies that the points are not collinear.

6. Given that: $A\ (1,-1)$,

$B\ (4,7)$, and

$P\ (h,-3)$

$$\frac{1-4}{h-4} = \frac{-1-7}{-3-7}$$

$$\frac{-3}{h-4} = \frac{-8}{-10}$$

$$-8h + 32 = 30$$

$$-8h = -2$$

$$h = \frac{1}{4}$$

30-38 Polar coordinates are a new topic for most students and care must be taken in their presentation. The primary difficulty is the multiplicity of the polar representations of a given point.

31 Other examples of the physical application of polar coordinates occur in air and sea navigation. The path of a racing sail boat beating up to a mark may appeal to some students. The paths across newly planted lawns on corner lots bear this out, too.

31 In the definition of the polar angle it may be necessary to stress that the terminal ray of the angle need not contain the point. This is a recurrent pitfall in verbal descriptions. The angle POM is not the only polar angle of the point P .

32 The fact that (r,θ) and $(-r,\theta + \pi)$ both represent the same point is worthy of emphasis. A student of the calculus must exercise particular care in the use of polar coordinates. If a curve is symmetric with respect to the origin, it is all too easy to sum up the area bounded by the curve on one side of the origin--and at the same time subtract away an equal area on the other. A judicious use of symmetry and boundaries is essential in such cases.

35 Once again we want to stress the freedom to choose our analytic framework in any way which will make algebraic manipulation as painless as possible. In general, if P and Q are any two distinct points in any plane and if (p_1,p_2) and (q_1,q_2) are any two distinct ordered pairs of real numbers, there exists a rectangular coordinate system in that plane in which $P = (p_1,p_2)$ and $Q = (q_1,q_2)$. Furthermore, if we let (r_1,θ_1) and (r_2,θ_2) be any two distinct ordered pairs of real numbers, there exists a polar coordinate system in the plane in which $P = (r,\theta_1)$ and $Q = (r_2,\theta_2)$. (Note that the change from (p_1,p_2) and (q_1,q_2) to (r_1,θ_1) and (r_2,θ_2) was

unnecessary; any two distinct ordered pairs of real numbers may be coordinates of P and Q in coordinate systems of each type. If at least one of the points is not on an axis, the coordinate system is unique.)

35 A moment's thought should convince you that the usual equations relating polar and rectangular coordinates are completely dependent upon a particular orientation of both coordinate systems in the same plane. If either coordinate system should be introduced differently into the plane, we would have to develop new equations of transformation.

36 The ordered pairs (r,θ) satisfying equations (2) describe two distinct points, but once the student has developed some facility with polar coordinates, it will be easy to choose the appropriate ones. If the students are familiar with the inverse trigonometric relations, they may prefer some equivalent of the follwoing definition,

$$P = \{(r,\theta) : \text{ where } r = \pm\sqrt{x^2 + y^2} \neq 0 \text{ , } \theta = \cos^{-1}\frac{x}{r}$$

$$= \sin^{-1}\frac{y}{r} \text{ ;}$$

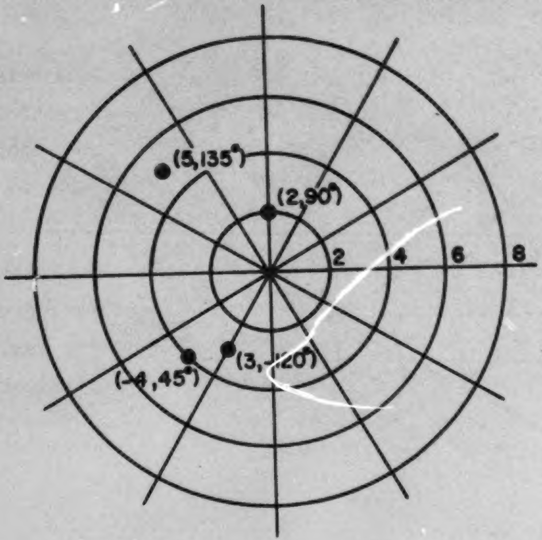
where $x^2 + y^2 = 0$, $r = 0$ and θ is any real number.} Hopefully, a student will ask what to do when $x = 0$, since one of the equations of transformation is not defined. Some other student should be able to point out that in this case $\theta = \frac{\pi}{2} + n\pi$, where n is any integer.

37 Example 5 is worth some attention, for the application of the Law of Cosines as a distance formula in polar coordinates is often convenient. Again there is a loophole, for it may not be apparent that the Law of Cosines still applies if $\theta_1 = \theta_2 + n\pi$, where n is any integer. In Section 2-7 we shall have occasion to point out that the relationship described still holds even when the "vertices of the triangle" are collinear.

38-40 There is a wealth of practice exercises here. Exercise 5 would require seventy different answers if all parts were done; Exercise 10 has over thirty answers. You will probably want to pick and choose within this set of exercises, but there is plenty of extra drill available for students who need it.

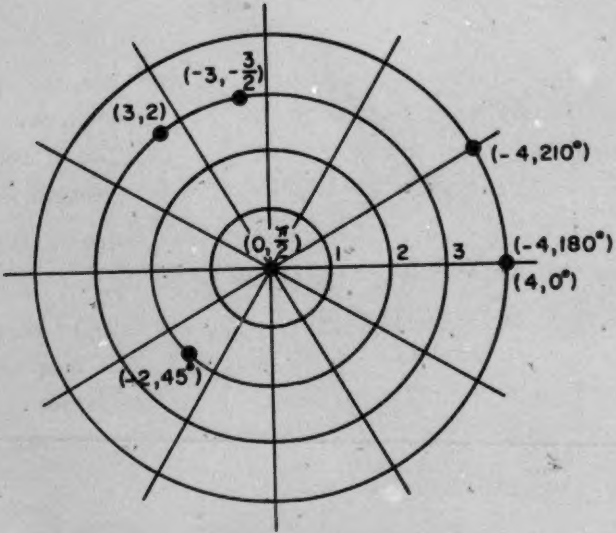
Exercises 2-4

1.

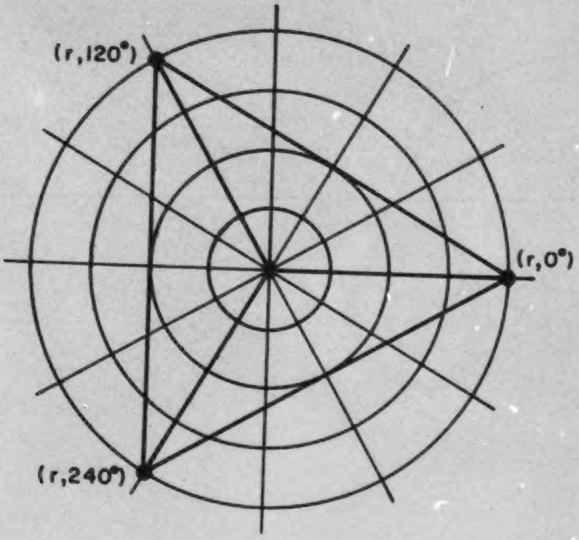

(5,135°)
(2,90°)
2
4
6
8
(-4,45°)
(3,-120°)

(5,135°)(-5,315°)
(5,495°)
(-5,-45°)
(2,90°)(-2,270°)
(2,450°)
(-2,-90°)
(-4,45°)(4,-135°)
(4,225°)
(-4,405°)
(3,-120°) (-3,60°)
(3,240°)
(3,600°)

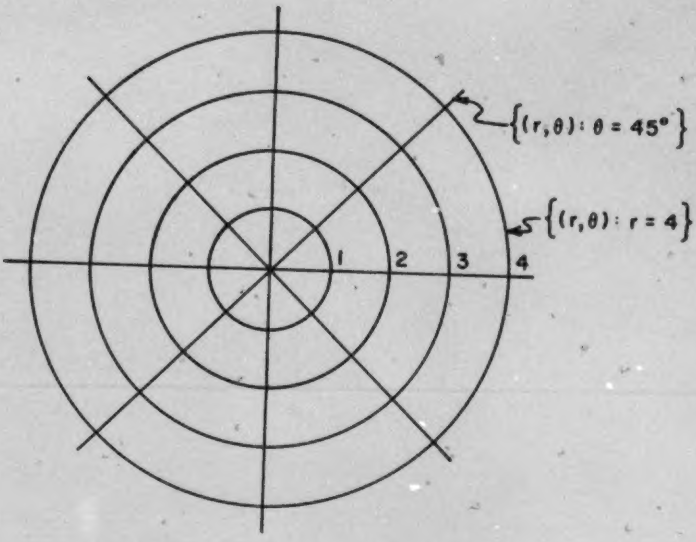
2.


(-3,-3/2)
(3,2)
(-4,210°)
(0,π/2)
1
2
3
(-4,180°)
(4,0°)
(-2,45°)

3.


(r,120°)
(r,0°)
(r,240°)

4.


{(r,θ): θ = 45°}
{(r,θ): r = 4}
1
2
3
4

5.	A$(2,270^\circ)$	$(2,-90^\circ)$	$(-2,90^\circ)$	$(2,\frac{3\pi}{2})$	$(-2,90^\circ)$
	B$(3,300^\circ)$	$(3,-60^\circ)$	$(-3,120^\circ)$	$(3,\frac{5\pi}{3})$	$(-3,120^\circ)$
	C$(4,330^\circ)$	$(4,-30^\circ)$	$(-4,150^\circ)$	$(4,\frac{11\pi}{6})$	$(-4,150^\circ)$
	D$(5,0^\circ)$	$(5,-0^\circ)$	$(-5,180^\circ)$	$(5,0)$	$(5,0^\circ)$
	E$(6,30^\circ)$	$(6,-330^\circ)$	$(-6,210^\circ)$	$(6,\frac{\pi}{6})$	$(6,30^\circ)$
	F$(7,60^\circ)$	$(7,-300^\circ)$	$(-7,240^\circ)$	$(7,\frac{\pi}{3})$	$(7,60^\circ)$
	G$(8,90^\circ)$	$(8,-270^\circ)$	$(-8,270^\circ)$	$(8,\frac{\pi}{2})$	$(8,90^\circ)$
	H$(9,120^\circ)$	$(9,-240^\circ)$	$(-9,300^\circ)$	$(9,\frac{2\pi}{3})$	$(9,120^\circ)$
	I$(10,150^\circ)$	$(10,-210^\circ)$	$(-10,330^\circ)$	$(10,\frac{5\pi}{6})$	$(10,150^\circ)$
	J$(\frac{3}{2},180^\circ)$	$(\frac{3}{2},-180^\circ)$	$(-\frac{3}{2},0^\circ)$	$(\frac{3}{2},\pi)$	$(-\frac{3}{2},0^\circ)$
	K$(\frac{5}{2},210^\circ)$	$(\frac{5}{2},-150^\circ)$	$(-\frac{5}{2},30^\circ)$	$(\frac{5}{2},\frac{7\pi}{6})$	$(-\frac{5}{2},30^\circ)$
	L$(\frac{7}{2},240^\circ)$	$(\frac{7}{2},-120^\circ)$	$(-\frac{7}{2},60^\circ)$	$(\frac{7}{2},\frac{4\pi}{3})$	$(-\frac{7}{2},60^\circ)$
	M$(5,285^\circ)$	$(5,-75^\circ)$	$(-5,105^\circ)$	$(5,\frac{19\pi}{12})$	$(-5,105^\circ)$
	N$(6,315^\circ)$	$(6,-45^\circ)$	$(-6,135^\circ)$	$(6,\frac{7\pi}{4})$	$(-6,135^\circ)$

6. (a) $(0,0)$
   (b) $(1,-1)$
   (c) $(\frac{5}{2},\frac{5}{2}\sqrt{3})$
   (d) $(4,0)$
   (e) $(-1,0)$
   (f) $(0,\sqrt{2})$
   (g) $(-1,-\sqrt{3})$
   (h) $(\sqrt{2},-\sqrt{2})$

7. (a) $(\sqrt{2},45^\circ)$
   (b) $(2\sqrt{2},315^\circ)$
   (c) $(p,0^\circ)$
   (d) $(q,\frac{\pi}{2})$
   (e) $(2,150^\circ)$
   (f) $(2,240^\circ)$
   (g) $(\sqrt{29},22^\circ)$
   (h) $(\sqrt{17},166^\circ)$

8. (a) $d(A,B)$ when $A = (2,150^\circ)$ and $B = (4,210^\circ)$

$$= \sqrt{(2)^2 + (4)^2 - 2(2)(4)\cos(210^\circ - 150^\circ)} = 2\sqrt{3}$$

Using rectangular coordinates

$A = (2,150^\circ)$ in rectangular coordinates $(-\sqrt{3},1)$

$B = (4,210^\circ)$ in rectangular coordinates $(-2\sqrt{3},-2)$

$$d(A,B) = \sqrt{(-\sqrt{3} - (-2\sqrt{3}))^2 + (1 - (-2))^2}$$

$$= \sqrt{(\sqrt{3})^2 + (3)^2} = 2\sqrt{3}$$

(b) Using rectangular coordinates:

$A = (5,\frac{5}{4}\pi)$ in rectangular coordinates $(-\frac{5}{2}\sqrt{2},-\frac{5}{2}\sqrt{2})$

$B = (12,\frac{7}{4}\pi)$ in rectangular coordinates $(6\sqrt{2},-6\sqrt{2})$

9. (a) $d(A,B) = \sqrt{34}$

(b) $A = (2,37^\circ)$, $B = (3,100^\circ)$

$d(A,B) = \sqrt{4 + 9 - 2(2)(3)\cos(100 - 37)}$

$d(A,B) = \sqrt{4 + 9 - 12(.454)}$

$d(A,B) = \sqrt{4 + 9 - 5.45} = \sqrt{7.55} = 2.75$

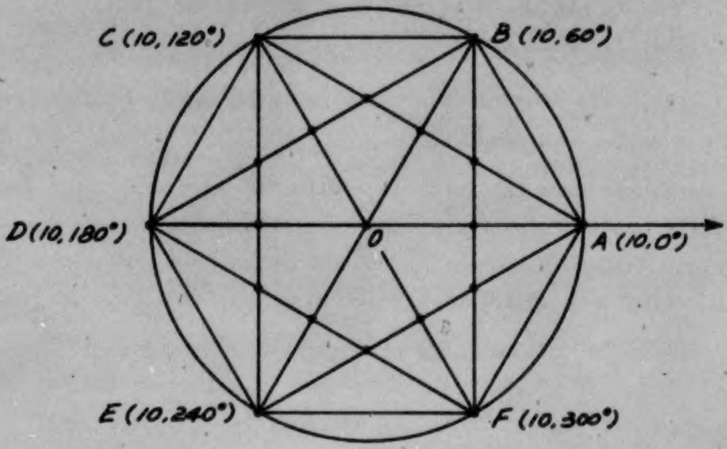
(c) $d(A,B) = \sqrt{52}$

(d) $d(A,B) = \sqrt{7}$

(e) $d(A,B) = 7$

(f) $d(A,B) = 5\sqrt{5}$

10.



	$\overline{DF}$	$\overline{CF}$	$\overline{CE}$	$\overline{BF}$	$\overline{BE}$	$\overline{BD}$	$\overline{AD}$	$\overline{AC}$
$\overline{AC}$	--	C	C	$(\frac{10}{3}\sqrt{3}, 30^\circ)$	$(5, 60^\circ)$	$(\frac{10}{3}\sqrt{3}, 90^\circ)$	A	
$\overline{AD}$	D	O	$(5, 180^\circ)$	$(5, 0^\circ)$	O	D		
$\overline{AE}$	$(\frac{10}{3}\sqrt{3}, 270^\circ)$	$(5, 300^\circ)$	E	$(\frac{10}{3}\sqrt{3}, 330^\circ)$	E	--		
$\overline{BD}$	D	$(5, 120^\circ)$	$(\frac{10}{3}\sqrt{3}, 150^\circ)$	B	B			
$\overline{BE}$	$(5, 240^\circ)$	O	E	B				
$\overline{BF}$	F	F	--					
$\overline{CE}$	$(\frac{10}{3}\sqrt{3}, 210^\circ)$	C						
$\overline{CF}$	F							
$\overline{DF}$								

This chart shows the points of intersection of the diagonals of a hexagon inscribed in circle with radius 10 one vertex at $(10, 0^\circ)$.

The twelve interior points of intersection different from 0 are

$(\frac{10}{3}\sqrt{3}, 30^\circ)$	$(\frac{10}{3}\sqrt{3}, 90^\circ)$	$(\frac{10}{3}\sqrt{3}, 150^\circ)$
$(\frac{10}{3}\sqrt{3}, 210^\circ)$	$(\frac{10}{3}\sqrt{3}, 270^\circ)$	$(\frac{10}{3}\sqrt{3}, 330^\circ)$
$(5, 0^\circ)$	$(5, 60^\circ)$	$(5, 120^\circ)$
$(5, 180^\circ)$	$(5, 240^\circ)$	$(5, 300^\circ)$

11. (a) $((-1)^k r_0, (\theta_0 + 180k)^\circ)$
    (b) $((-1)^k r_0, \theta_0 + \pi k)$

41-49 Students should find little if any new material in this section. It is included for review and completeness.

41 The geometric form is useful in developing equations for a line, since it is closely allied both to the geometric picture and, since the denominators are direction numbers for the line, to the parametric representation for the line. It corresponds to the symmetric equations for a line in 3-space.

43 Inclination is defined geometrically, since our point of view is geometric. This definition may also prepare the student for the definition of direction angles in the following section.

44 Note that inclination is defined even when slope is not.

49 Since the general form of an equation of a line does not reveal immediately the geometric characteristics of the line, it is worthwhile to develop facility in interpreting the geometric properties from the coefficients.

Exercises 2-5

1. $y + 3 = 2(x - 2)$ $2x - y - 7 = 0$ $p = 7$ $q = 3$
2. $y - 5 = -\frac{2}{3}(x + 3)$ $p = -6$ $q = -\frac{1}{3}$
3. $y = 3x + b$ $p = \frac{7 - b}{3}$ $q = 15 + b$
4. $y - 5 = \frac{2}{3}(x - 4)$ The two lines are parallel.
5. $y = k(x - a)$ y-intercept at $(0, -ka)$
6. $ax + by = 0$ a, b real numbers.
   $5x + 3y = 0$ contains $(-3, 5)$
7. Slope of $\overleftrightarrow{OA}$ is $\frac{5}{3}$ slope of $\overleftrightarrow{OB}$ is $-\frac{3}{5}$

   Two lines are perpendicular if and only if
   (a) the product of their slopes is -1 or
   (b) one has no slope and the other zero slope.
8. $\frac{x + 8}{4} = \frac{y - 8}{-3}$

9. (1) $\frac{x + 4}{6} = \frac{y - 8}{-5}$ (5) $\frac{y}{\frac{14}{3}} + \frac{x}{\frac{28}{5}} = 1$

(2) $5x + 6y - 28 = 0$

(3) $y - 8 = -\frac{5}{6}(x + 4)$ (6) $y - 8 = \frac{3 - 8}{2 + 4}(x + 4)$

(4) $y = -\frac{5}{6}x + \frac{14}{3}$ (7) $x + 4 = \frac{2 + 4}{3 - 8}(y - 8)$

Slope: $-\frac{5}{6}$ x-intercept: $\frac{28}{5}$ y-intercept: $\frac{14}{3}$

$$y = -\frac{a}{b}x - \frac{c}{b}$$

10. (a) If $b = 0$, $ac \neq 0$, line is vertical, through $(-\frac{c}{a}, 0)$

(b) If $a = 0$, $bc \neq 0$, line is horizontal, through $(0, -\frac{c}{b})$

(c) If $c = 0$, $ab \neq 0$, line has slope $-\frac{a}{b}$, through $(0,0)$

11. (a) $y = -\frac{7}{3}x + 5$

(b) $y = x - 5$

(c) $y = -\frac{2}{7}x + \frac{17}{7}$

(d) $y = -x - 2$

(e) $y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3} - 9}{3}$

12. (a) $\frac{x - 3}{1 - 3} = \frac{y - 2}{-2 - 2}$

(b) The midpoint of $\overline{BC}$ is $(\frac{3}{2}, 3)$

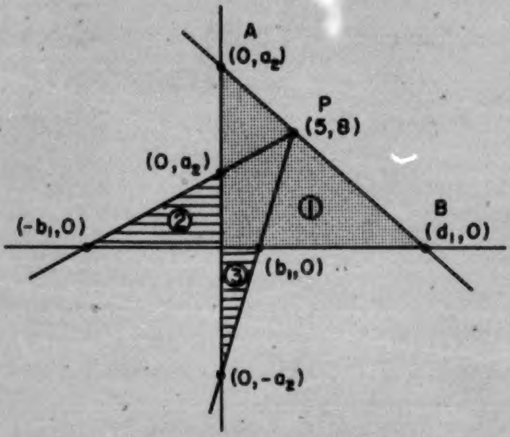
Median from A can be represented by

$$\frac{x - 1}{\frac{3}{2} - 1} = \frac{y - (-2)}{3 - (-2)}, \text{ or } 10x - y - 12 = 0 .$$

(c) The midpoint of $\overline{AC}$ is $(\frac{1}{2}, 1)$. And from (b) midpoint of $\overline{BC}$ is $(\frac{3}{2}, 3)$. Line joining these two points is represented by

$$\frac{x - \frac{1}{2}}{\frac{3}{2} - \frac{1}{2}} = \frac{y - 1}{3 - 1}, \text{ or } 2x - y = 0 .$$

13. Given the conditions of the problem, it appears that there are three possible solutions. (sketch below)



Triangle ①: This triangle is not satisfactory, since its area must be greater than 40 ; that is, its area includes that of the rectangle with 0 and P as opposite vertices, and adjacent sides on the axes.

Triangle ②: The area of the triangle is $\frac{1}{2}a_2b_1$. The slope of $\overline{BP}$ = slope of $\overline{AB}$ and

$$\frac{8}{5 + b_1} = \frac{a_2}{b_1}$$

Solving for a_2 ,

$$a_2 = \frac{8b_1}{5 + b_1}$$

Substituting into $\frac{1}{2}a_2b_1$, we find that the positive root is $5 (b_1 = 5)$.

Using $a_2 = \frac{8b_1}{5 + b_1}$, we find $a_2 = 4$.

The equation of the line through (0,4) , (-5,0) , and (5,8) using the symmetric form is

$$\frac{x + 5}{0 + 5} = \frac{y - 0}{4 - 0} \text{, or } 4x - 5y + 20 = 0 .$$

Triangle ③: Area of triangle 3 is $\frac{1}{2}a_2 b_1$.

Slope of $\overline{PB}$ = slope of $\overline{AB}$

$$\frac{8}{5 - b_1} = \frac{a^2}{b_1}$$

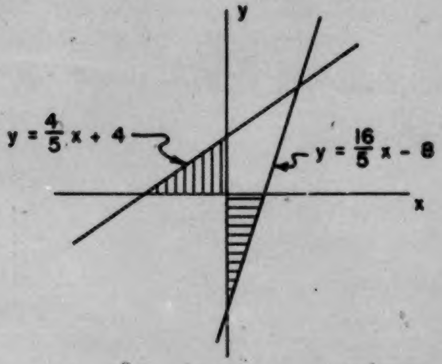
Solving for a_2 , we see that

$$a_2 = \frac{8b_1}{5 - b_1}$$

Substituting in area formula, $b_1 = \frac{20}{8}$ and $a_2 = 8$.

The equation of the line through (0,-8) , $(\frac{20}{8}, 0)$ and (5,8) in symmetric form is

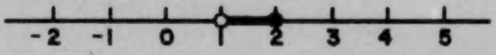
$$\frac{x - 0}{5 - 0} \quad \frac{y - (-8)}{8 - (-8)}, \text{ or } 16x - 5y - 40 = 0 .$$



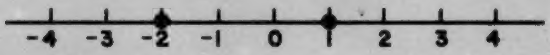
Since this is such a long chapter, you may want to test the students at this point. With this in mind we have included a copious set of review and challenge exercises from which selections may be made.

Review Exercises - Section 2-1 through Section 2-5

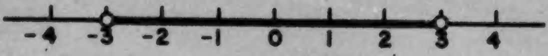
1. $\{x:\ 1 < x \le 2\}$



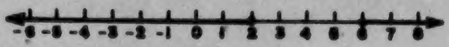
2. $\{x:\ (x - 1)(x + 2) = 0\}$



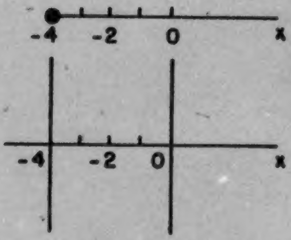
3. $\{x: |x| < 3\}$



4. $\{x: |x - 4| \geq 2\}$

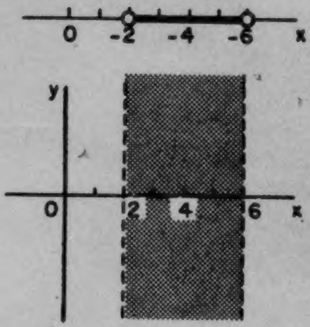

-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8

5. One-space: A point four units to the left of the origin.
Two-space: A line parallel to the y-axis four units to the left of it.

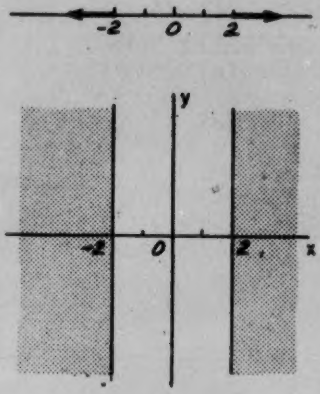


6. The empty set.

7. One-space: A segment of the x-axis between, but not including the points $x = 2$ and $x = 6$.
2-space: A portion of the xy-plane between but excluding lines $x = 2$ and $x = 6$.

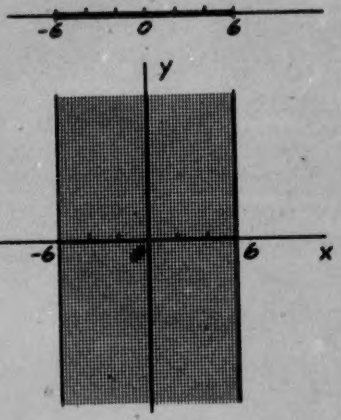

0
-2
-4
-6
x
y
0
2
4
6
x

8. One-space: The portion of the x-axis to the right of 2 including $x = 2$ and to the left of and including $x = -2$.
2-space: The portion of the plane to the right of and including the line $x = 2$ and the portion of the plane to the left of and including the line $x = -2$.



9. One-space: A segment of the x-axis between and including the points $x = 6$ and $x = -6$.

2-space: The portion of the plane between and including the lines $x = 6$ and $x = -6$.



10. Let m represent the midpoints and t_1, t_2 represent the trisection points.

(a) $m = \frac{1}{2}$

$t_1 = 0$ and $t_2 = 1$

(b) $m = -2$

$t_1 = -3$ and $t_2 = -1$

(c) $m = \frac{1}{2}$

$t_1 = -\frac{1}{3}$ and $t_2 = 1\frac{1}{3}$

11. (a) $(2, \frac{\pi}{3})$

(b) $(2, \frac{3\pi}{4})$

(c) $(5, -53^\circ)$, approximately.

(d) $(\sqrt{13}, 236^\circ)$, approximately

(e) $(1, 0)$

(f) $(1, \frac{\pi}{2})$

12. (a) $(2\sqrt{2}, 2\sqrt{2})$

(b) $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$

(c) $(\sqrt{2}, \sqrt{2})$

(d) $(3\sqrt{2}, 3\sqrt{2})$

(e) $(\frac{-5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{2})$

(f) $(\frac{-3\sqrt{3}}{2}, \frac{3}{2})$

13. $3x + 4y = 14$

14. $8x - 11y + 46 = 0$

15. $5x - 2y + 10 = 0$

16. $y = \sqrt{3}x + 5 - 4\sqrt{3}$

17. $y = 6$

18. $x = 4$

19. The equation for $\overleftrightarrow{AB}$ is $y = -\sqrt{3}\ x + 6\sqrt{3}$

The equation for $\overleftrightarrow{BC}$ is $y = 3\sqrt{3}$

The equation for $\overleftrightarrow{CD}$ is $y = \sqrt{3}\ x + 6\sqrt{3}$

The equation for $\overleftrightarrow{DE}$ is $y = -\sqrt{3}\ x - 6\sqrt{3}$

The equation for $\overleftrightarrow{EF}$ is $y = -3\sqrt{3}$

The equation for $\overleftrightarrow{FA}$ is $y = \sqrt{3}\ x - 6\sqrt{3}$

20. The equation for $\overleftrightarrow{AB}$ is $\sqrt{3}\ x + y - 6\sqrt{3} = 0$

The equation for $\overleftrightarrow{BC}$ is $y - 3\sqrt{3} = 0$

The equation for $\overleftrightarrow{CD}$ is $\sqrt{3}\ x - y + 6\sqrt{3} = 0$

The equation for $\overleftrightarrow{DE}$ is $\sqrt{3}\ x + y + 6\sqrt{3} = 0$

The equation for $\overleftrightarrow{EF}$ is $y + 3\sqrt{3}\ y = 0$

The equation for $\overleftrightarrow{FA}$ is $\sqrt{3}\ x - y - 6\sqrt{3} = 0$

21. The equation for $\overleftrightarrow{AB}$ is $\dfrac{x - 6}{-3} = \dfrac{y}{3\sqrt{3}}$

The equation for $\overleftrightarrow{BC}$ is not defined

The equation for $\overleftrightarrow{CD}$ is $\dfrac{x + 6}{3} = \dfrac{y}{3\sqrt{3}}$

The equation for $\overleftrightarrow{DE}$ is $\dfrac{x + 3}{-3} = \dfrac{y + 3\sqrt{3}}{3\sqrt{3}}$

The equation for $\overleftrightarrow{EF}$ is not defined

The equation for $\overleftrightarrow{FA}$ is $\dfrac{x - 3}{3} = \dfrac{y + 3\sqrt{3}}{3\sqrt{3}}$

22. $\dfrac{-\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{AC}$.

$\dfrac{\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{BD}$.

$\dfrac{\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{AE}$.

$\dfrac{-\sqrt{3}}{3}$ is the slope of $\overleftrightarrow{DF}$.

23. Let t_1 and t_2 represent the trisection points.

For $\overline{AB}$, $t_1 = (5,\sqrt{3})$ and $t_2 = (4,2\sqrt{3})$.

For $\overline{BC}$, $t_1 = (1,3\sqrt{3})$ and $t_2 = -1,3\sqrt{3})$.

For $\overline{CD}$, $t_1 = (-4,2\sqrt{3})$ and $t_2 = (-5,\sqrt{3})$.

For $\overline{DE}$, $t_1 = (-5,-\sqrt{3})$ and $t_2 = (-4,-2\sqrt{3})$.

For $\overline{EF}$, $t_1 = (-1,-3\sqrt{3})$ and $t_2 = (1,-3\sqrt{3})$.

For $\overline{FA}$, $t_1 = (4,-2\sqrt{3})$ and $t_2 = (5,-\sqrt{3})$.

24. (a) $P = (4,2\sqrt{3})$ or $(8,-2\sqrt{3})$

(b) $Q = (\frac{3}{7}, 3\sqrt{3})$ or $(21,3\sqrt{3})$

(c) $R = (-\frac{13}{3}, \frac{5\sqrt{3}}{3})$ or $(9,15\sqrt{3})$

25. The inclination of $\overleftrightarrow{AB} = 120^\circ$

The inclination of $\overleftrightarrow{AC} = 150^\circ$

The inclination of $\overleftrightarrow{AE} = 30^\circ$

The inclination of $\overleftrightarrow{AF} = 60^\circ$

26.

Symmetric form.	displays direction pair	does not exist for lines parallel to either axis
General form.	always exists displays direction pair ease in computing intersections ease in telling if L contains (0, 0)	conceals intercepts
Point-slope form.	displays slope ease in testing if P is on L	does not always exist
Slope-intercept form.	displays slope and intercept	does not always exist
Intercept form.	displays intercepts displays a direction pair	does not always exist
Two-point form.	usual way of finding line through two points determines slope	must be used in different form if $\overleftrightarrow{P_1P_2}$ is vertical

(a) general form

(b) intercept form

(c) general form

(d) slope-intercept form

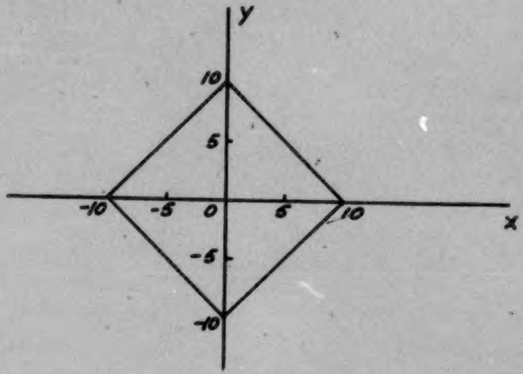
(e) slope-intercept

(f) symmetric

(g) symmetric

(h) symmetric

27. A square as shown in the figure



28. It is interesting to have students note what happens as the constant term shrinks to zero. At this instant the square shrinks to a point. The teacher might ask what happens when the constant is negative.

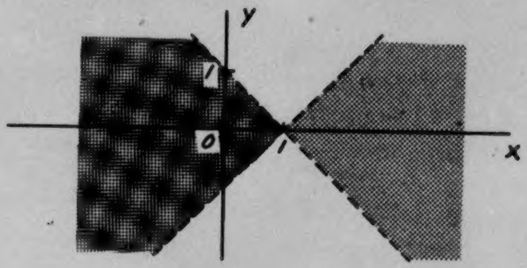
29. The half-plane above and excluding the line $x - y = 1$.

30. The half-plane above and including the line $x - y = 1$.

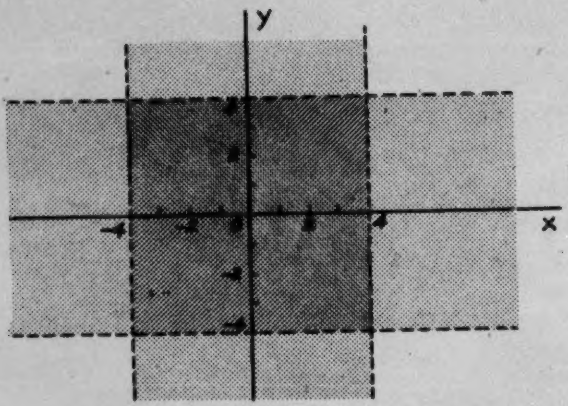
31. The "triangular" portion of the plane below and excluding the lines $x - y = 1$ and $x + y = 1$.

Graph for Exercise 17.

Cross hatch shows intersection set



32. The graph of R_1 in 2-space is the vertical strip of the plane between and excluding the lines $x = -4$ and $x = 4$.



The graph of R_2 in 2-space is the horizontal strip of the plane between and excluding lines $y = 4$, $y = -4$. The cross-hatch in the graph represents $R_1 \cap R_2$.

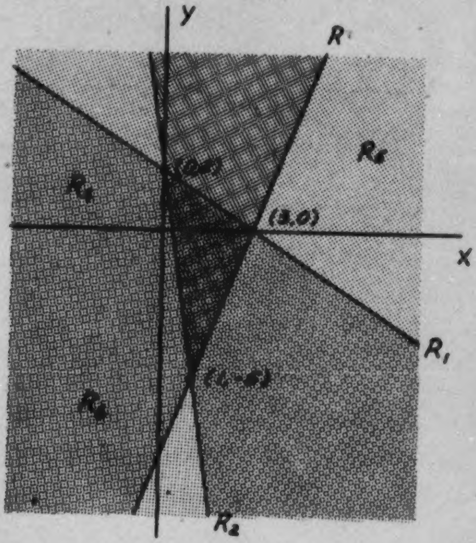
In one-space R_1 is a segment between and excluding points $x = 4$ and $x = -4$; for R_2 the same situation prevails on the y-axis. (The line for points y may be any line.) $R_1 \cap R_2$ is a single point, provided the x-axis intersects the y-axis.

In 3-space we can visualize R_1 and R_2 as the path of the 2-space graph for each separate set as it moves perpendicular to the plane of the page; $R_1 \cap R_2$ as a rectangular solid perpendicular to the plane of the page. The bounding planes are excluded from the graphs.

33. If $<$ is replaced by $\leq$ the graphs would be as in Exercise 18 except the boundaries would be included in every case. For $R_1 \cup R_2$ apply definition of union of sets. The instructor may very well use this group of exercises as an informal introduction to families of curves. Note the role of the parameter.

34. Use two-point or point-slope or otherwise to obtain $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$. Science students need not memorize the formula; they can derive it.

35. The separate graphs R_1 to R_6 are labeled in the figure. $R_4 \cap R_5 \cap R_6$ is the set of all points on the triangle and its interior as shown by the cross hatch.



Challenge Exercises

1 Good students should enjoy this confrontation with ideas that go beyond the routine.

(a) Set of lines parallel to y-axis through points $(x, 0)$ where x ranges over the integers.

(b) Set of lines parallel to the x-axis through points $(0, y)$ where y ranges over the integers.

(c) The set of all lattice points of the plane.

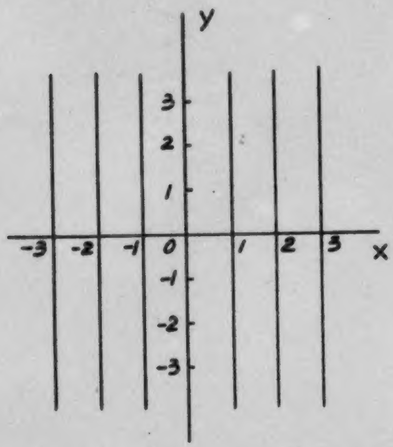
(d) Includes all of R_1, R_2, R_3. A grill such as paper ruled in cross section.

(e) Boundaries on the heavy sides are included.

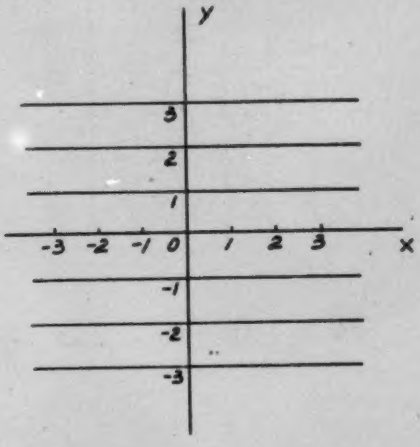
(f) Same graph moved k units to the right.

(g) and (h) Notice effect of placement of minus signs.

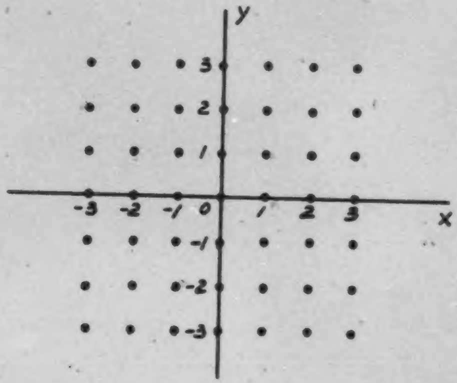
(a) $R_1 = \{(x, y): [x] = x\}$



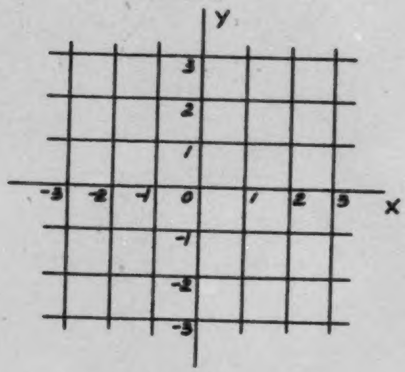
(b) $R_2 = \{(x, y): [y] = y\}$



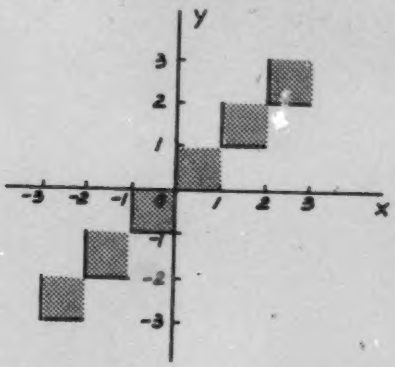
(c) $R_3 = \{(x, y):[x] = x\} \cap \{(x, y):[y] = y\}$



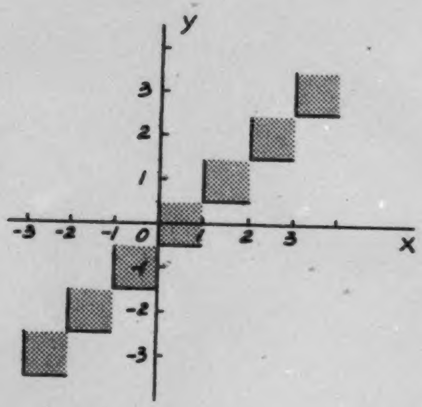
(d) $R_4 = R_1 \cup R_2$



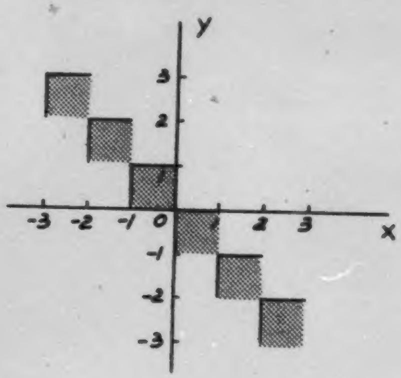
(e) $R_5 = \{(x, y):[x] = [y]\}$



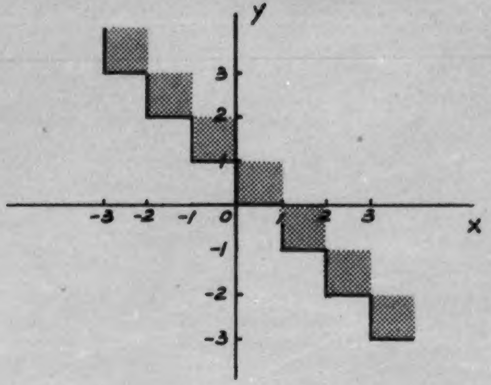
(f) $R_6 = \{(x, y):[x] = [y + k]\}$



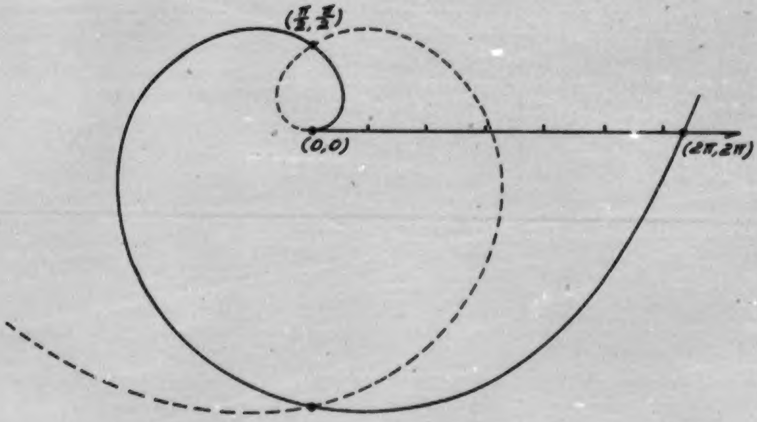
(g) $R_7 = \{(x, y):[x] = [-y]\}$



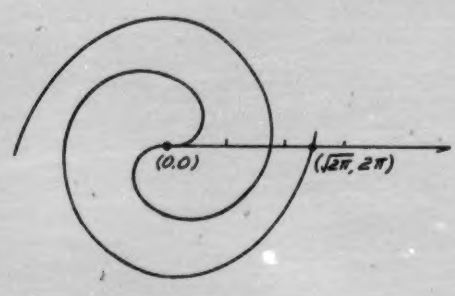
(h) $R_8 = \{(x, y): [x] = -[y]\}$



2. $\{(r, \theta): r = \theta\}$ [dotted line accounts for negative values of r]



3. $\{(r, \theta): r^2 = \theta\}$



4. (a) $d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (\frac{r}{s})^2(y_2 - y_1)^2}$

(b) $d(P_1,P_2) = \sqrt{(\frac{s}{r})^2(x_2 - x_1)^2 + (y_2 - y_1)^2}$

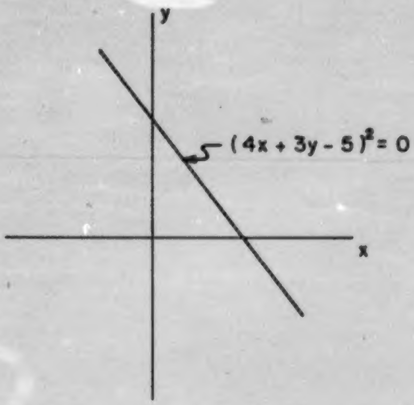
(c) $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ must either be parallel or have supplementary inclinations.

Let $\alpha = \frac{r}{s}$. From part (a) we know that for $d(P,Q) = d(R,S)$ we must have $(p_1 - q_1)^2 + \alpha^2(p_2 - q_2)^2 = (r_1 - s_1)^2 + \alpha^2(r_2 - s_2)$.

But also $(p_1 - q_1)^2 + (p_2 - q_2)^2 = (r_1 - s_1)^2 + (r_2 - s_2)^2$.

Thus $(1 - \alpha^2)(p_2 - q_2)^2 = (1 - \alpha^2)(r_2 - s_2)^2$. Since $r \neq s$, we know $\alpha^2 \neq 1$. Therefore, $1 - \alpha^2 \neq 0$ and we may divide by $1 - \alpha^2$. From the result we see that the distances in the y-direction must be equal. But then the distances in the x-direction must be equal. These conditions are satisfied only when $\overline{PQ}$ and $\overline{RS}$ are parallel or when they have supplementary inclinations.

5.

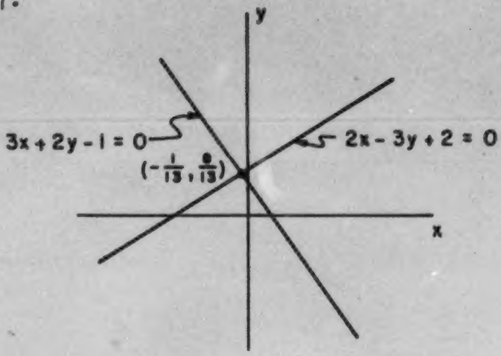


The line may be written in a simpler analytic representation.

$$4x + 3y - 5 = 0 .$$

6. The graph of $(ax + by + c)^k = 0$ is the same as the graph of $ax + by + c = 0$. A simpler representation is $ax + by + c = 0$.

7.


y
x
3x + 2y - 1 = 0
2x - 3y + 2 = 0
$(-\frac{1}{13}, \frac{8}{13})$

$$3x + 2y - 1 = 0$$

$$y = -\frac{3}{2}x + \frac{1}{2}$$

$$2x - 3y + 2 = 0$$

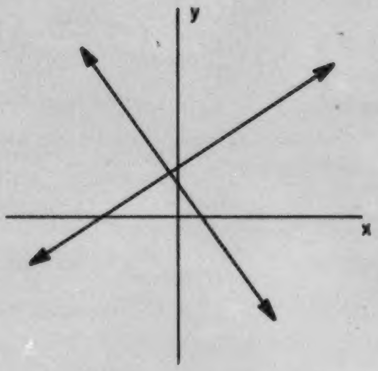
$$y = \frac{2}{3}x + \frac{2}{3}$$

$$-\frac{3}{2}x + \frac{1}{2} = \frac{2}{3}x + \frac{2}{3}$$

$$x = -\frac{1}{13}$$

$$y = \frac{8}{13}$$

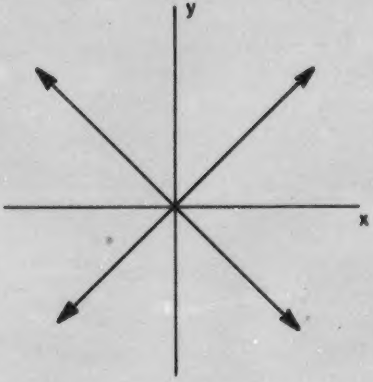
8.



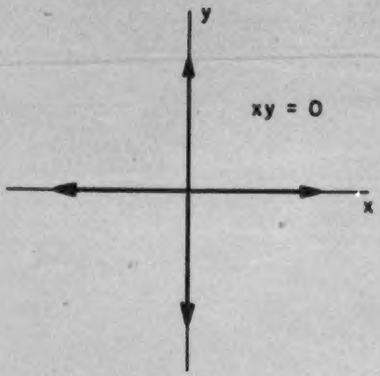
$$(3x + 2y - 1)(2x - 3y + 2) = 0$$

9.

$$(x + y)(x - y) = 0$$



10.



11. (a) rational
    (b) rational
    (c) real
    (d) complex

12. (a) R may be any line containing the point $(-\frac{4}{5}, -\frac{1}{5})$ except

$$L = \{(x,y) : x + y + 1 = 0\} .$$

(b) S may be any line containing the point $(-\frac{4}{5}, -\frac{1}{5})$ except

$$L = \{(x,y) : 3x - 2y + 2 = 0\} .$$

(c) T may be any line containing the point $(-\frac{4}{5}, -\frac{1}{5})$.

13. (a) U is the whole plane except for the points of

$$L = \{(x,y) : x + y + 1\}$$

other than $(-\frac{4}{5}, -\frac{1}{5})$.

(b) V is the whole plane except for the points of

$$L = \{(x,y) : 3x - 2y + 2 = 0\}$$

other than $(-\frac{4}{5}, -\frac{1}{5})$

(c) W is the whole plane.

14. There are two possibilities: $L_0 = \{(x,y) : a_0x + b_0y + c_0 = 0\}$ and $L_1 = \{(x,y) : a_1x + b_1y + c_1 = 0\}$ may intersect at a point (x_0,y_0) . In this case,

(a) R may be any line containing (x_0,y_0) except L_1 ,

(b) S may be any line containing (x_0,y_0) except L_0 ,

(c) T is the whole plane except those points of L_1 other than (x_0,y_0) ,

(d) U is the whole plane except those points of L_0 other than (x_0,y_0) ,

(e) V may be any line containing (x_0,y_0) , and

(f) W is the whole plane.

L_0 and L_1 may be parallel. In this case,

(a) unless R is empty, it is a line parallel to L_0 and L_1 except L_1 , when $k = 0$, $R = L_0$; when $0 < k$ R is between L_0 and L_1 ; when $-1 < k < 0$, L_0 is between L_1 and R ; when $k = -1$, R is empty (the null set); when $k < -1$, L_1 is between L_0 and R .

(b) The same argument holds for S , but the roles of L_0 and L_1 are reversed.

(c) T is the whole plane except L_1 .

(d) U is the whole plane except L_0 .

(e) unless V is empty, it is a line parallel to L_0 and L_1 . When $n = 0$, $V = L_0$; when $m = 0$, $V = L_0$.

(f) W is the whole plane.

15. (a) the null (or empty) set.
(b) the whole plane.

We include a copious set of Illustrative Test Items from which we may wish to make selections.

Illustrative Test Items for Sections 2-1 through 2-5

1. If P and Q have coordinates 3 and -5 respectively in one linear coordinate system on the line and corresponding coordinates -2 and 3 respectively in a second linear coordinate system, what are the corresponding coordinates of points with the following coordinates in the first coordinate system?

   (a) 0
   (b) 1
   (c) -1
   (d) $-\frac{1}{5}$
   (e) $-1\frac{4}{5}$
   (f) -13
   (g) 11
   (h) 10

2. If M , A , and B are the midpoint and trisection points of $\overline{PQ}$, find m , a , and b when

   (a) $p = 3$, $q = 12$
   (b) $p = -3$, $q = 1$
   (c) $p = -2$, $q = 13$
   (d) $p = 2r + 3s$, $q = 3r - 2s$

3. If the coordinates of P , Q , and R are 2 , x , and 12 respectively, find the value(s) of x such that

   (a) $d(P,Q) = \frac{1}{5} d(P,R)$
   (b) $d(P,R) = 2d(P,Q)$
   (c) $d(P,Q) = 5d(P,R)$
   (d) $d(P,Q) = 2d(R,P)$
   (e) $d(Q,P) = \frac{1}{2}d(P,R)$

4. If M , A , and B are the midpoint and trisection points of $\overline{PQ}$, find the coordinates of M , A , and B when

   (a) $P = (2,1)$, $Q = (-4,-2)$
   (b) $P = (7,1)$, $Q = (-2,1)$
   (c) $P = (-2,5)$, $Q = (7,12)$
   (d) $P = (p_1,p_2)$, $Q = (q_1,q_2)$
   (e) $P = (1,r)$, $Q = (s + r,2s - 3)$

5. P , Q , and R are points in a plane with a rectangular coordinate system. Determine whether the three points are collinear if

   (a) $P = (-5,5)$, $Q = (0,0)$, $R = (7,-7)$
   (b) $P = (-1,5)$, $Q = (8,-3)$, $R = (-7,-6)$
   (c) $P = (1,2)$, $Q = (9,10)$, $R = (-3,-2)$
   (d) $P = (9,-10)$, $Q = (-8,5)$, $R = (0,-2)$

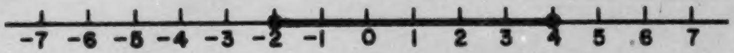
6. A line with slope $-\frac{2}{3}$ passes through $(-3,4)$. If the points $(p,7)$ and $(5,q)$ are on the line, find p and q .

7. Sketch the graphs of the sets of points on a line with the following analytic representations.

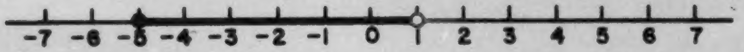
   (a) $[x:\ -1 \leq x < 4]$
   (b) $[x:\ |x - 5| < 2]$
   (c) $[x:\ (x - 1)(x - 3) \leq 0]$
   (d) $[x:\ x(x + 2)(x - 3) = 0]$

8. Find analytic conditions which describe the illustrated sets of points.

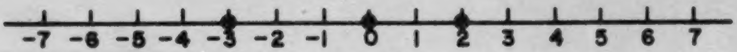
(a)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

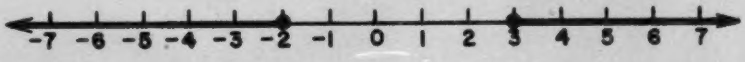
(b)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

(c)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

(d)


-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

9. Find three polar representations for the point with rectangular coordinates

   (a) $(3,3\sqrt{3})$
   (b) $(-2,-2)$
   (c) $(-1,\sqrt{3})$
   (d) $(-2\sqrt{3},-2)$
   (e) $(4,-4)$
   (f) $(1,-\frac{1}{\sqrt{3}})$
   (g) $(6,0)$
   (h) $(0,-12)$

10. Find rectangular coordinates for the point with polar coordinates

(a) $(4,0)$
(b) $(\sqrt{2},45^\circ)$
(c) $(6,-120^\circ)$
(d) $(5,\frac{5\pi}{6})$
(e) $(-3,-\frac{3\pi}{4})$
(f) $(-4,-\frac{11\pi}{6})$

11. Without changing to rectangular coordinates find the distance between the points whose polar coordinates are

(a) $(5,0)$ and $(12,\frac{\pi}{2})$
(b) $(6,0)$ and $(6,-\pi)$
(c) $(4,45^\circ)$ and $(5,-135^\circ)$
(d) $(3,\frac{\pi}{3})$ and $(4,\frac{2\pi}{3})$
(e) $(-6,-\frac{\pi}{4})$ and $(5,\frac{\pi}{4})$
(f) $(-3,-90^\circ)$ and $(6,90^\circ)$

12. Find an equation in the indicated form for the line which

(a) contains $(5,3)$ and $(6,4)$; symmetric form.
(b) contains $(0,4)$ and $(3,0)$; intercept form.
(c) contains $(7,-6)$, slope $-\frac{2}{3}$; point-slope form.
(d) contains $(13,-6)$ and $(-2,12)$; general form .
(e) contains $(0,-5)$, slope $\frac{3}{2}$; slope-intercept form.
(f) contains $(9,10)$ and $(-\sqrt{2},4)$; two-point form.
(g) contains $(-5,12)$, inclination $\frac{3\pi}{4}$; point-slope form.
(h) contains $(5,7)$ and $(5,-3)$; two-point form.
(i) contains $(3,-6)$ and $(-3,3)$; intercept form.
(j) x-intercept 2 ; y-intercept 4 ; general form.
(k) x-intercept 5 ; inclination 60° ; slope-intercept form.
(l) contains $(-5,7)$, slope $\frac{6}{7}$; symmetric form.
(m) contains $(-5,-4)$, inclination 45° ; general form.
(n) contains $(7,-2)$, slope $\frac{7}{13}$; symmetric form.

(o) contains $(6,-5)$ and $(-3,2)$; two-point form.

(p) contains $(3,4)$, slope -2 ; intercept form.

(q) contains $(6,1)$ and $(-2,5)$; slope-intercept form.

(r) contains $(9,3)$ and $(9,12)$; general form.

(s) contains $(2,3)$ and $(-7,3)$; general form.

(t) contains $(-5,4)$, inclination $\frac{2\pi}{3}$; point-slope form.

13. Show that the triangle ABC is a right triangle if $A = (-1,-3)$, $B = (11,8)$, and $C = (-3,4)$.

14. Find an equation in general form of the line containing the median to side $\overline{BC}$ of triangle ABC if $A = (-2,7)$, $B = (3,4)$, and $C = (1,-2)$.

15. Find the area of the triangle determined by the lines

$$L_1 = \{(x,y):\ 2x - 8 = 0\} ,$$
$$L_2 = \{(x,y):\ 12x - 5y - 53 = 0\} ,$$
$$L_3 = \{(x,y):\ 4x - 5y + 19 = 0\} .$$

16. In triangle ABC , $A = (0,0)$, $B = (6,0)$ and $C = (0,8)$.
    - (a) The bisector $\angle A$ divides the segment $\overline{BC}$ in what ratio?
    - (b) The point D at which the bisector of $\angle A$ intersects $\overline{BC}$?
    - (c) Find $d(B,D)$ and $d(C,D)$.

17. Find the coordinates of the points in which the line that contains $(-8,3)$ and $(3,-2)$ intersects the axes.

Answers

1. (a) $-\frac{1}{8}$
   (b) $-\frac{3}{4}$
   (c) $\frac{1}{2}$
   (d) 0
   (e) 1
   (f) 8
   (g) -7
   (h) $-6\frac{3}{8}$

2. (a) $m = 7\frac{1}{2}$, $a = 6$,

(b) $m = -1$, $a = -1\frac{2}{3}$, $b = -\frac{1}{3}$

(c) $m = 5\frac{1}{2}$, $a = 3$, $b = 8$

(d) $m = \frac{5r + s}{2}$, $a = \frac{7r + 4s}{3}$, $b = \frac{8r - s}{3}$

3. (a) 0 , 4
(b) -3 , 7
(c) -48 , 52
(d) -18 , 22
(e) -3 , 7

4. (a) $M = (-1, -\frac{1}{2})$ $A = (0,0)$ $B = (-2,-1)$

(b) $M = (2\frac{1}{2}, 1)$ $A = (4,1)$ $B = (1,1)$

(c) $M = (2\frac{1}{2}, 8\frac{1}{2})$ $A = (1, 7\frac{1}{3})$ $B = (4, 9\frac{2}{3})$

(d) $M = \left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}\right)$ $A = \left(\frac{2p_1 + q_1}{3}, \frac{2p_2 + q_2}{3}\right)$ $B = \left(\frac{p_1 + 2q_1}{3}, \frac{p_2 + 2q_2}{3}\right)$

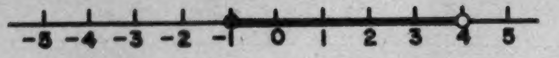
(e) $M = \left(\frac{r + s + 1}{2}, \frac{r + 2s - 3}{2}\right)$ $A = \left(\frac{r + s + 2}{3}, \frac{2r + 2s - 3}{3}\right)$ $B = \left(\frac{2r + 2s + 1}{3}, \frac{r + 4s - 6}{3}\right)$

5. (a) Yes
(b) No
(c) Yes
(d) No

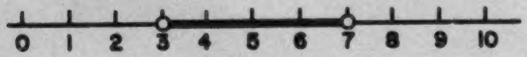
(Determine the distances between the pairs of points; the points are collinear if and only if the sum of the two shorter distances equals the longer. More simply, use slopes; the points are collinear if and only if the slope of $\overline{PQ}$ equals the slope of $\overline{PR}$.)

6. $p = -7\frac{1}{2}$, $q = -\frac{4}{3}$.

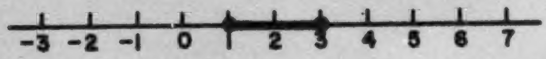
7. (a)



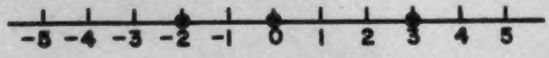
(b)



(c)



(d)



8. (a) $\{x: \ -2 \le x \le 4\}$, $\{x: \ |x - 1| \le 3\}$, $\{x: \ (x + 2)(x - 4) \le 0\}$, or the equivalent.

(b) $\{x: \ -5 \le x < 1\}$, $\{x: \ \frac{x - 1}{x - 1} |x + 2| \le 3\}$, $\{x: \ \frac{x - 1}{x - 1}(x+5)(x-1) \le 0\}$, or the equivalent.

(c) $\{x: \ x(x + 3)(x - 2) = 0\}$,$\{-3,0,2\{$, or the equivalent.

(d) $\{x: \ x \le -2 \ \text{or} \ x \ge 3\}$, $\{x: \ |x - \frac{1}{2}| \ge 2\frac{1}{2}\}$, $\{x: \ (x + 2)(x - 3) \ge 0\}$, or the equivalent.

9. (There are, of course, unlimited possibilities for the answers to this question; we give only a few.)

(a) $(6,\frac{\pi}{3})$, $(-6,\frac{4\pi}{3})$, $(6,60^\circ)$, $(-6,240^\circ)$.

(b) $(2\sqrt{2},\frac{5\pi}{4})$, $(-2\sqrt{2},\frac{\pi}{4})$, $(2\sqrt{2},225^\circ)$, $(-2\sqrt{2},45^\circ)$

(c) $(2,\frac{2\pi}{3})$, $(-2,\frac{5\pi}{3})$, $(2,120^\circ)$, $(-2,300^\circ)$

(d) $(4,\frac{7\pi}{6})$, $(-4,\frac{\pi}{6})$, $(4,210^\circ)$, $(-4,30^\circ)$

(e) $(4\sqrt{2},\frac{7\pi}{4})$, $(-4\sqrt{2},\frac{3\pi}{4})$, $(4\sqrt{2},315^\circ)$, $(-4\sqrt{2},135^\circ)$

(f) $(\frac{2}{\sqrt{3}},\frac{\pi}{6})$, $(-\frac{2}{\sqrt{3}},\frac{7\pi}{6})$, $(\frac{2}{\sqrt{3}},30^\circ)$, $(-\frac{2}{\sqrt{3}},210^\circ)$

(g) $(6,0)$, $(-6,\pi)$, $(6,0^\circ)$, $(-6,180^\circ)$

(h) $(12,\frac{3\pi}{2})$, $(-12,\frac{\pi}{2})$, $(12,270^\circ)$, $(-12,90^\circ)$

10. (a) $(4,0)$
(b) $(1,1)$
(c) $(-3,-3\sqrt{3})$
(d) $(-\frac{5\sqrt{3}}{2}, 2\frac{1}{2})$
(e) $(\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}})$
(f) $(-2\sqrt{3},2)$

11. (a) 13
(b) 12
(c) 9
(d) $\sqrt{13}$
(e) $\sqrt{61}$
(f) 3

12. (a) $\frac{x - 5}{6 - 5} = \frac{y - 3}{4 - 3}$
(b) $\frac{x}{3} + \frac{y}{4} = 1$
(c) $y + 6 = -\frac{2}{3}(x - 7)$
(d) $6x + 5y - 48 = 0$
(e) $y = \frac{3}{2}x - 5$
(f) $y - 10 = \frac{4 - 10}{-\sqrt{2} - 9}(x - 9)$
(g) $y - 12 = -1(x + 5)$
(h) $x - 5 = \frac{5 - 5}{-3 - 7}(y - 7)$
(i) $\frac{x}{-1} + \frac{y}{-\frac{3}{2}} = 1$
(j) $4x - 2y - 8 = 0$
(k) $y = \sqrt{3}\,x - 5\sqrt{3}$
(l) $\frac{x + 5}{2 + 5} = \frac{y - 7}{13 - 7}$
(m) $x - y + 1 = 0$
(n) $\frac{x - 7}{20 - 7} = \frac{y + 2}{5 + 2}$
(o) $y + 5 = \frac{2 + 5}{-3 - 6}(x - 6)$
(p) $\frac{x}{5} + \frac{y}{10} = 1$
(q) $y = -\frac{1}{2}x + 4$
(r) $x = 9$
(s) $y = 3$
(t) $y - 4 = -\sqrt{3}(x + 5)$

13. (a) $\left(d(A,B)\right)^2 = (-1 - 11)^2 + (-3 - 8)^2 = 265$

$\left(d(B,C)\right)^2 = (11 + 3)^2 + (8 - 4)^2 = 212$

$\left(d(A,C)\right)^2 = (-1 + 3)^2 + (-3 - 4)^2 = 53$

Since $\left(d(A,B)\right)^2 = \left(d(B,C)\right)^2 + \left(d(A,C)\right)^2$, by the converse of the Pythagorean Theorem triangle ABC is a right triangle with $\angle ACB$ the right angle.

(b) If you permit students to use the fact that the product of the slopes is -1 if and only if lines are perpendicular, the proof follows more readily from the fact that

$$m_{\overline{AC}} \cdot m_{\overline{BC}} = \left(-\frac{7}{2}\right)\cdot\left(\frac{2}{7}\right) = -1 .$$

14. $3x + 2y - 8 = 0$

15. 20

16. (a) 3 to 4

(b) $\left(3\frac{3}{7}, 3\frac{3}{7}\right)$

(c) $d(B,D) = 4\frac{2}{7}$; $d(C,D) = 5\frac{5}{7}$

17. The line intersects the x-axis at $\left(-\frac{7}{5}, 0\right)$;

the line intersects the y-axis at $\left(0, -\frac{7}{11}\right)$.

57-63 Most students will probably believe they have a clear intuitive understanding of the idea of the two directions on a line and may feel the discussion here is pointless. As with the notion of a directed segment, it may help to ask them to try to explain what they mean accurately, using terms with clear geometric meanings. When they find that this is not at all easy, they may be convinced that our approach is worth studying.

57 The open question of lines without slope is considered in Exercise 5 on page 64. At this point we assume that the student recalls that parallel, nonvertical lines have the same slope. In Section 2-7 we shall reafirm this fact.

57 We shall use the idea of equivalent direction numbers for a line a great deal; if a student does not grasp this idea now, he may find it a frequent stumbling block.

58 You may well note that had we chosen directed angles to describe the lines in the plane, a single angle would suffice. However, a pair of nonnegative angles is conventional and leads to symmetric representation; it is also desirable, since a triple of direction angles is much neater in 3-space. The extension to spaces of higher dimension is immediate with the approach adopted here.

59-60 The fact that the pair of normalized direction numbers and the pair of direction cosines are equal is extremely convenient.

61 The context which specifies a direction for a line varies and is, of course, frequently quite colloquial, as "the line from P to Q".

Exercise 6 on page 64 asks for a justification that the alternative direction angles for a line are respectively supplementary.

62 The information developed in the solution to Example 4(b) is quite useful. The student should develop facility in extracting from a general form of an equation of a line direction numbers and direction cosines for the line.

63 The importance of Example 5 may not be apparent. It provides what little initfal motivation there is for the normal form of the equation of a line.

64 Exercise 7 might well be discussed briefly even if it is not assigned, for it develops a relationship which is useful in relating the equations of a line in polar and rectangular coordinates.

Exercises 2-6

1. (a) $(-3,4)$ or $(3,-4)$
   (b) $(4,1)$ or $(-4,-1)$
   (c) $(0,6)$ or $(0,-6)$
   (d) $(-5,0)$ or $(5,0)$
   (e) $(1,1)$ or $(-1,-1)$
   (f) $(2,2)$ or $(-2,-2)$
   (g) $(-1,1)$ or $(1,-1)$
   (h) $(-4,4)$ or $(4,-4)$

2. (a) $(-\frac{3}{5}, \frac{4}{5})$ or $(\frac{3}{5}, -\frac{4}{5})$

   (b) $(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}})$ or $(\frac{-4}{\sqrt{17}}, \frac{-1}{\sqrt{17}})$

   (c) $(0,1)$ or $(0,-1)$

   (d) $(-1,0)$ or $(1,0)$

   (e) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   (f) $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   (g) $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

   (h) $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

3. (a) $\alpha = 127^\circ$, $\beta = 37^\circ$; or $\alpha = 53^\circ$, $\beta = 143^\circ$ (approximately)

   (b) $\alpha = 76^\circ$, $\beta = 14^\circ$; or $\alpha = 104^\circ$, $\beta = 166^\circ$ (approximately)

   (c) $\alpha = 90^\circ$, $\beta = 0^\circ$; or $\alpha = 90^\circ$, $\beta = 180^\circ$

   (d) $\alpha = 180^\circ$, $\beta = 90^\circ$; or $\alpha = 0^\circ$, $\beta = 90^\circ$

   (e) $\alpha = 45^\circ$, $\beta = 45^\circ$; or $\alpha = 135^\circ$, $\beta = 135^\circ$

   (f) $\alpha = 45^\circ$, $\beta = 45^\circ$; or $\alpha = 135^\circ$, $\beta = 135^\circ$

   (g) $\alpha = 135^\circ$, $\beta = 45^\circ$; or $\alpha = 45^\circ$, $\beta = 135^\circ$

   (h) $\alpha = 135^\circ$, $\beta = 45^\circ$; or $\alpha = 45^\circ$, $\beta = 135^\circ$

4. (a)

$(3,-4)$	$(2,0)$	$(0,-3)$	$(-1,2)$	$(-2,1)$
$-\frac{4}{3}$	0	not defined	-2	$-\frac{1}{2}$

   (b) $(\frac{3}{5}, -\frac{4}{5})$, or any equivalent given by $(\frac{3c}{5}, -\frac{4c}{5})$, $c \neq 0$.

   $\alpha = 53^\circ$, $\beta = 143^\circ$; or $\alpha = 127^\circ$, $\beta = 37^\circ$

   $(1,0)$, or any equivalent given by $(c,0)$, $c \neq 0$.

   $\alpha = 0^\circ$, $\beta = 90^\circ$; or $\alpha = 180^\circ$, $\beta = 90^\circ$

   $(0,-1)$, or any equivalent given by $(0,-c)$, $c \neq 0$.

   $\alpha = 90^\circ$, $\beta = 180^\circ$; or $\alpha = 90^\circ$; $\beta = 0^\circ$

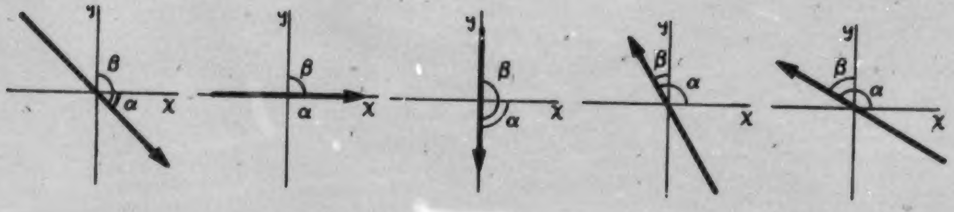
   $(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, or any equivalent given by $(-\frac{c}{\sqrt{5}}, \frac{2c}{\sqrt{5}})$, $c \neq 0$.

   $\alpha = 117^\circ$, $\beta = 27^\circ$; or $\alpha = 63^\circ$, $\beta = 153^\circ$

   $(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, or any equivalent given by $(-\frac{2c}{\sqrt{5}}, \frac{c}{\sqrt{5}})$, $c \neq 0$.

   $\alpha = 153^\circ$, $\beta = 63^\circ$; or $\alpha = 27^\circ$, $\beta = 117^\circ$

(c) and (d)



5. A pair of direction numbers determined by P_0 and P_1 are

$(\ell_1, m_1) = (0, y_1 - y_0)$; $m_1 = y_1 - y_0 \neq 0$, $\ell_1 = 0$.

A pair of direction numbers determined by P_0 and P_2 are

$(\ell_2, m_2) = (0, y_2 - y_0)$; $m_2 = y_2 - y_0 \neq 0$ and $\ell_2 = 0$.

Since $m_1 \neq 0$ and $m_2 \neq 0$, both

$$c_1 = \frac{m_2}{m_1} \quad \text{and} \quad c_2 = \frac{m_1}{m_2}$$

are defined and not equal to zero. Thus,

$$(c_1\ell_1 , c_1m_1) = (\ell_2 , m_2) \quad \text{and} \quad (c_2\ell_2 , c_2m_2) = (\ell_1 , m_1) .$$

$$(0, y_1 - y_0) \quad \text{and} \quad (0, y_2 - y_0)$$

are equivalent pairs of direction numbers for the vertical line.

6. $\cos \alpha = \dfrac{\ell}{\sqrt{\ell^2 + m^2}}$ $\qquad$ $\cos \beta = \dfrac{m}{\sqrt{\ell^2 + m^2}}$

$\cos \alpha' = \dfrac{\ell}{\sqrt{\ell^2 + m^2}}$ $\qquad$ $\cos \beta' = \dfrac{-m}{\sqrt{\ell^2 + m^2}}$

So $\cos \alpha' = - \cos \alpha$ $\qquad$ $\cos \beta' = - \cos \beta$

Hence $\alpha' = \pm \alpha + p\pi$ $\qquad$ $\beta' = \pm \beta + q\pi$ $\quad$ p, q odd integers but α, α', β, and β' are between 0 and π, so the only solutions are $\beta' + \beta = \pi$, $\alpha' + \alpha = \pi$.

7. (a) 1. In the Figure 2-13a , $\omega = \frac{\pi}{2} - \beta + 2\pi n$. Therefore

$\sin \omega = \sin (\frac{\pi}{2} - \beta)$ but since the sine of an angle is equal to the cosine of its complement,

$$\sin \omega = \cos \beta$$

2. In Figure 2-13b , $\omega = \beta - \frac{\pi}{2} + 2\pi n$. Therefore

$$\sin \omega = \sin (\beta - \frac{\pi}{2})$$

$$\sin \omega = \sin [-(\frac{\pi}{2} - \beta)]$$

$$\sin \omega = \cos (-\beta)$$

$$\sin \omega = \cos \beta$$

3. In Figure 2-13c, $\omega + \frac{\pi}{2} + \beta = 180 + 2\pi n$ and $\omega = \frac{\pi}{2} - \beta + 2\pi n$.

The result is the same as part 1 above.

4. In Figure 2-13d, $\omega - \beta = \frac{\pi}{2} + 2\pi n$ and $\omega = \frac{\pi}{2} + \beta + 2\pi n$.

Therefore $\sin \omega = \sin (\frac{\pi}{2} + \beta)$

$$\sin \omega = \sin \frac{\pi}{2} \cos \beta + \cos \frac{\pi}{2} \sin \beta$$

Since $\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$

$$\sin \omega = \cos \beta$$

(b) 1. If the positive ray lies on the positive half of the x-axis,

$$\omega = 2\pi n \text{ and } \beta = \frac{\pi}{2}$$

Since we wish to show that $\sin \omega = \cos \beta$, we may substitute and see that

$$\sin 2\pi n = \cos \frac{\pi}{2} = 0$$

2. If the positive ray lies on the positive half of the y-axis,

   $\omega = \frac{\pi}{2} + 2\pi n$ and $\beta = 0$ and $\sin \frac{\pi}{2} = \cos 0 = 1$

3. If the positive ray lies on the negative half of the x-axis,

   $\omega = \pi + 2\pi n$ and $\beta = \frac{\pi}{2}$ and $\sin \pi = \cos \frac{\pi}{2} = 0$

4. If the positive ray lies on the negative half of the y-axis,

   $\omega = \frac{3\pi}{2} + 2\pi n$, $\beta = \pi$ and $\sin \frac{3\pi}{2} = \cos \pi = -1$

8. (a) $(-2,2)$

$\left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

$\alpha = 153^\circ$ $\beta = 117^\circ$

(b) $(-2,1)$

$\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$\alpha = 153^\circ$, $\beta = 63^\circ$

(c) $(6,5)$

$\left(\frac{6}{\sqrt{61}}, \frac{5}{\sqrt{61}}\right)$

$\alpha = 40^\circ$, $\beta = 50^\circ$

65 It is traditional to talk about the angle between two lines, but present standards of precision require that we take account of the fact that at least four angles are formed when two lines intersect. These angles can be distinguished in a diagram by various methods, but all of these methods must induce a sense along each of the lines.

67-68 The second solution to Example (2) is given as a suggestion to the student that once he has recognized the form of the equations of the lines normal to a given line, he may write immediately the equation of the normal containing a given point.

68 Sometimes the results of our analytic approach describe additional situations not usually approached in the same way geometrically. The situation here furnishes a nice example of this.

69 Example 3(b) is also offered to show the student how he may use an equation of a given line in general form to write immediately an equation of a parallel line containing a given point.

70-71 Since $(b_1,-a_1)$ and $(b_2,-a_2)$ are pairs of direction numbers for the lines L_1 and L_2 respectively, we also note that

$$\cos\theta = \frac{\ell_1\ell_2 + m_1m_2}{\ell_2^2 + m_2^2 \sqrt{\ell_2^2 + m_2^2}}$$

or

$$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 .$$

In Exercise 12 on page 74 the student is asked to develop this relationship. It has some merit when the lines forming $\angle\theta$ are directed lines. In this case $\angle\theta$ is the angle formed by positive rays of L_1 and L_2 with endpoints at the point of intersection (if any) of L_1 and L_2. Exercise 15 on page 87 also calls for such an interpretation.

71-72 Example 5 is really a lemma to be used in the development of the normal form of an equation of a line in the following section.

Exercises 2-7

1. (a) $d(A,C) = d(B,C) + d(A,B)$, by the definition of betweenness for points. This is equivalent to

$$d(A,B) = d(A,C) - d(B,C) ,$$

which implies

$$\big(d(A,B)\big)^2 = \big(d(A,C)\big)^2 + \big(d(B,C)\big)^2 - 2d(A,C)\,d(B,C) ;$$

since $\cos C = \cos 0^\circ = 1$, we may write

$$\big(d(A,B)\big)^2 = \big(d(A,C)\big)^2 + \big(d(B,C)\big)^2 - 2d(A,C)\,d(B,C)\cos C$$

(b) Here we have

$$d(A,B) = d(A,C) + d(B,C) ,$$

which implies

$$\big(d(A,B)\big)^2 = \big(d(A,C)\big)^2 + \big(d(B,C)\big)^2 + 2d(A,C)\ d(B,C) ;$$

since $\cos C = \cos 180^\circ = -1$, we may write

$$\big(d(A,B)\big)^2 = \big(d(A,C)\big)^2 + \big(d(B,C)\big)^2 - 2d(A,C)\ d(B,C) \cos C .$$

2. (a) Equation (6) states that

$$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\ \sqrt{a_2^2 + b_2^2}} .$$

Substituting into Equation (6) ,

$$\cos\theta = \frac{a_1a_2}{\sqrt{a_1^2}\ \sqrt{a_2^2 + b_2^2}}$$

$$\cos\theta = \frac{\pm a_2}{\sqrt{a_2^2 + b_2^2}} .$$

Let α be the inclination of L_2 . Then the measures of the angles θ between L_1 and L_2 are $90^\circ - \alpha$ and $90^\circ + \alpha$.

$\cos\theta = \cos(90^\circ - \alpha) = \cos 90^\circ \cos\alpha + \sin 90^\circ \sin\alpha = \sin\alpha$

or

$\cos\theta = \cos(90^\circ + \alpha) = \cos 90^\circ \cos\alpha - \sin 90^\circ \sin\alpha = -\sin\alpha$.

Also we have $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = -\dfrac{a_2}{b_2}$

$$b_2 \sin\alpha = -a_2 \cos\alpha ,$$

and $b_2^2 \sin^2\alpha = a_2^2 \cos^2\alpha = a_2^2 (1 - \sin^2\alpha)$.

This is equivalent to $\sin^2\alpha = \dfrac{a_2^2}{a_2^2 + b_2^2}$,

and

$$\sin\alpha = \frac{\pm a_2}{\sqrt{a_2^2 + b_2^2}} = \cos\theta$$

(b) $$\cos\theta = \frac{a_1 a_2}{\sqrt{a_1^2}\ \sqrt{a_2^2}} = \pm 1$$

$\theta = 0^\circ$ or 180°, which is the case for parallel lines.

3. L_2 and L_5 are the same lines

   L_1 and L_4 are the same lines

   L_3 is perpendicular to L_1 and L_4

4. (a) $\theta = 7^\circ$
   (b) $\theta = 90^\circ$
   (c) $\theta = 45^\circ$
   (d) $\theta = 83^\circ$
   (e) $\theta = 0^\circ$ (lines are parallel).
   (f) $\theta = 90^\circ$

5. The slope of OP is $\frac{b}{a}$ and the slope of OQ is $\frac{a}{-b}$.

   Since $m_{\overline{OP}} \cdot m_{\overline{OQ}} = -1$, $\overline{OP} \perp \overline{OQ}$.

6. (a) $2x - 3y = 0$
   (b) $3x + y - 8 = 0$
   (c) $3x + 2y - 17 = 0$
   (d) $x - 3y - 5 = 0$

7. (a) $2x - 5y + 31 = 0$
   (b) $2x - 3y + 17 = 0$
   (c) $16x - 6y - 13 = 0$
   (d) $y = 7$
   (e) $x = 5$

8. $D = (4,-8)$.

   3 possibilities; $(12,2)$ and $(-2,12)$ are the others.

9. The slope of L_1 is $\frac{4}{3}$

   $$y + 2 = \frac{4}{3}(x - 1)$$

10. (a) $\overleftrightarrow{AB}$: $2x + 7y - 17 = 0$

$\overleftrightarrow{BC}$: $x + y - 1 = 0$

$\overleftrightarrow{CA}$: $3x + 8y - 23 = 0$

(b) $m_{\overleftrightarrow{AB}} = -\frac{2}{7}$

$m_{\overleftrightarrow{BC}} = -1$

$m_{\overleftrightarrow{CA}} = -\frac{3}{8}$

(c)

$$m\angle CBA = 151^\circ$$

$$\cos\theta_1 = \frac{2 + 7}{\sqrt{1 + 1}\sqrt{4 + 49}} = .874$$

$$\theta_1 = 29^\circ$$

The angle desired is the supplement of θ_1 or $180^\circ - 29^\circ$ or 151°

$$m\angle BCA$$

$$\cos\theta_2 = \frac{3 + 8}{\sqrt{1 + 1}\sqrt{9 + 64}} = .910^+$$

$$\theta_2 = 24^\circ$$

$$m\angle CAB =$$

$$\cos\theta_3 = \frac{6 + 56}{\sqrt{4 + 49}\sqrt{9 + 64}} = .997^-$$

$$\theta_3 = 5^\circ$$

(d) Altitude to side $\overleftrightarrow{AB}$

$7x - 2y + 29 = 0$

Altitude to side $\overleftrightarrow{BC}$

$x - y - 4 = 0$

Altitude to side $\overleftrightarrow{AC}$

$8x - 3y + 25 = 0$

11. (a) $L_1' = \{(x,y):\ b_1x - a_1y = 0\}$

$L_2' = \{(x,y):\ b_2x - a_2y = 0\}$

(b) $\therefore \cos\theta = \dfrac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}$ and using Equation (6),

$$\cos \phi = \frac{b_1b_2 + a_1a_2}{\sqrt{(b_1)^2 + (-a)^2}\ \sqrt{b_2{}^2 + (-a_2)^2}}$$

$$\cos \theta = \cos \phi$$

If L_1' is $\perp$ to L_1 and L_2' is $\perp$ to L_2, then the measure of an angle between L_1 and L_2 is equal to the measure of an angle between L_1' and L_2'.

12. (a) $L_1 = \{(x,y)\colon\ \lambda_1x + \mu_1y + c_1 = 0\}$

$L_2 = \{(x,y)\colon\ \lambda_2x + \mu_2y + c_2 = 0\}$

$$\cos \theta = \frac{\lambda_1\lambda_2 + \mu_1\mu_2}{\sqrt{\lambda_1{}^2 + \mu_1{}^2}\ \sqrt{\lambda_2{}^2 + \mu_2{}^2}}$$

but $$\lambda_1{}^2 + \mu_1{}^2 = \lambda_2{}^2 + \mu_2{}^2 = 1$$

$$\cos \theta = \lambda_1\lambda_2 + \mu_1\mu_2$$

(b) If $\cos \theta$ is positive $0^\circ \leq \theta \leq 90^\circ$ and $\angle\theta$ is the least angle formed by L_1 and L_2.

(c) Assume $L_1 \perp L_2$

$$m_1 = -\frac{\lambda_1}{\mu_1} \text{ and } m_2 = -\frac{\lambda_2}{\mu_2} \text{ and}$$

$$m_1m_2 = -1 .$$

So $\left(-\frac{\lambda_1}{\mu_2}\right)\left(-\frac{\lambda_2}{\mu_2}\right) = -1$ and

$$\lambda_1\lambda_2 = -\mu_1\mu_2 \qquad \text{or}$$

$$\lambda_1\lambda_2 = -\mu_1\mu_2 = 0$$

Conversely assume $\lambda_1\lambda_2 + \mu_1\mu_2 = 0$

but $$\lambda_1\lambda_2 + \mu_2\mu_2 = 0 = \cos \theta$$

and $$\cos \theta = 0$$

$$\therefore \theta = 90^\circ \text{ and}$$

$$L_1 \perp L_2 .$$

75-78 The normal form of an equation of a line is troublesome to develop, for students have usually not considered the characterization of a line by a normal segment from the origin. Therefore, the argument for bothering to develop it at all must rest upon its applications; it is not at all a natural extension in the students' eyes. With this in mind, before beginning this section it might be helpful to challenge the students to find the distance between a line and a point not on the line. Once they have been forced to the trouble of finding (a) the slope of the perpendiculars to the given line, (b) an equation of the perpendicular containing the given point, (c) the point of intersection of this perpendicular and the given line, and (d) the distance between the point of intersection and the given point, they may be more in a mood to pursue a development which solves this problem more easily.

76 The conventional notation does lead to confusion here. It is easy for the student to confuse the coefficients in the normal form with the direction cosines of the line itself. Emphasis on the reason for the name "normal form" may shorten the period of confusion. Then, too, an oral drill on the following information to be gleaned from the normal form may help.

If $\lambda > 0$ and $\mu > 0$, the line extends above the origin from upper left to lower right; if $\lambda < 0$ and $\mu > 0$, above the origin from lower left to upper right; if $\lambda < 0$ and $\mu < 0$, below the origin from upper left to lower right; if $\lambda > 0$ and $\mu < 0$, below the origin from lower left to upper right. If $\lambda = 0$ and $\mu = 1$, the line is horizontal and above the origin; if $\lambda = 0$ and $\mu = -1$, horizontal and below the origin; if $\mu = 0$ and $\lambda = 1$, vertical and to the right of the origin; if $\mu = 0$ and $\lambda = -1$, vertical and to the left of the origin.

To make sense of this information a student will have to keep in mind that (λ, μ) is the pair of direction cosines of the normal segment.

77 The fact that authorities differ in the case of lines containing the origin has a backhanded sort of significance. There seems to be little reason to recognize a difference which does not make a difference. E.g., $1.\overline{0} = 0.\overline{9}$; there is no numerical difference.

78 If your students are already versed in the parametric representation of lines, there is a neater approach to the problem.

The line $\overleftrightarrow{FP_1}$ has the parametric representation

$$x = x_1 + \lambda t$$

$$y = y_1 + \mu t .$$

With this representation $|t|$ is the distance between (x,y) and $P_1 = (x_1,y_1)$. In particular, if we let $F = (x_0,y_0)$, for some t, F has a representation

$$x_0 = x_1 + \lambda t_1$$
$$y_0 = y_1 + \mu t_1 .$$

Then

$$x_0 - x_1 = \lambda t_1$$
$$y_0 - y_1 = \mu t_1 ,$$

and

$$d(P_1,F) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} = \sqrt{(\lambda t_1)^2 + (\mu t_1)^2}$$
$$= |t_1| \sqrt{\lambda^2 + \mu^2} = |t_1| .$$

Since the point $F = (x_0,y_0)$ satisfies the equation $\lambda x + \mu y - p = 0$, we have

$$\lambda(x_1 + \lambda t_1) + \mu(y_1 + \mu t_1) - p = 0 ,$$

which is equivalent to

$$\lambda x_1 + \mu y_1 - p = -(\lambda^2 + \mu^2)\, t_1 = -t_1 .$$

Thus,

$$d(P_1,F) = |t_1| = |\lambda x_1 + \mu y_1 - p| .$$

With this approach we do not have to consider the five different cases.

79-82 The amount of classroom explication necessary on the polar form will depend upon the students' background in analytic trigonometry. Some familiarity with the addition formulas is essential. These are developed in SMSG Intermediate Mathematics, pages 605-610, and, of course, in any standard trigonometry text.

79 At this point you may wish to consider that since $P = (-r, \theta + \pi)$, the line also has the polar representation

$$-r \cos \left(\theta + (\pi - \omega)\right) = p .$$

This opens a question to which we shall return in Chapter 5, when we consider related polar equations.

80 Although the polar angle which contains the normal segment to L is the same set of points as the direction angle α and $\angle\omega = \angle\alpha$, our conventions for measuring these angles are different. The measure of $\angle\omega$

may be any real number, while $0 \leq \alpha \leq \pi$ (or $0 \leq \alpha \leq 180^\circ$) . Thus, even if we choose an $\angle\omega$ such that $|\omega|$ is minimal, we still are assured only that $|\omega| = \alpha$, or $\omega = \pm \alpha$. However, since $\omega = \pm \alpha + 2\pi n$ for any integer n, in the case we describe, we do have $\cos \omega = \cos (2\pi n \pm \alpha) = \cos \alpha$. The test should read $\omega = \pm \alpha + 2\pi n$ for any integer n .

81 Students may not be familiar with the technique of "normalizing" coefficients in order to rewrite

$$a \cos \theta + b \sin \theta \text{ as } \sqrt{a^2 + b^2} \sin(\theta + \alpha_1) .$$

where

$$\sin \alpha_1 = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \cos \alpha_1 = \frac{b}{\sqrt{a^2 + b^2}} ,$$

or as $\sqrt{a^2 + b^2} \cos (\theta - \beta_1)$, where

$$\cos \beta_1 = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin \beta_1 = \frac{b}{\sqrt{a^2 + b^2}} .$$

Therefore, you may wish to consider other examples than Part (e) of Example 5.

84-85 In assigning exercises you may well wish to consider Exercises 7 through 9, which suggest a further application of the normal form, and Exercises 12 through 17, which furnish practice in transforming equations from representations in one coordinate system to the other.

These last exercises open questions which will be considered in detail in Chapters 5 and 6. In the algebraic manipulation of polar equations we may frequently do some rather wild things which would get us into trouble in rectangular representations. The freedom we exploit stems from three considerations:

i) the multiplicity of the polar representations of a point,
ii) related polar equations, (See Chapter 5.)
iii) "factoring" equations. (See Chapter 6.)

For example, in Exercise 13 we suggest multiplication of both members of the equation by r . In rectangular representations such multiplication by a factor containing a variable is quite likely to add points to the graph, but here the points $(0,\theta)$, which might be added, are already included by the original representation as $(0,(n + \frac{1}{2}) \pi)$, where n is any integer.

In Exercise 12 we first obtain

$$r^2 = 36 \text{ , or } r^2 - 36 = (r - 6)(r + 6) = 0 .$$

Now the equations obtained by setting the factors of the left member equal to zero,

$$r = 6 \text{ and } r = -6 ,$$

are related polar equations (as defined on page 167 of the text), for they each have the same graph as $r^2 = 36$. Since each is a simpler representation of the graph, later on we shall prefer either one to the first equation.

In Exercise 17 we first obtain

$$(r^2 + r \sin \theta)^2 = r^2 .$$

If we divide both members by r^2 , we obtain

$$(r + \sin \theta)^2 = 1 ,$$

but we have not lost any points from the graph. The pole is the only point we might have lost, and it is still represented by

$$(0,(n + \tfrac{1}{2})\pi) ,$$

where n is any integer. Then we may factor to obtain

$$(r + \sin \theta - 1)(r + \sin \theta + 1) = 0 ;$$

the equations

$$r = 1 - \sin \theta \text{ and } r = -(1 + \sin \theta) ,$$

which are suggested by the factors of the original equation, are related polar equations. Their graphs are identical to the graph of the original equation, and either one is a far simpler representation.

In summary, multiplication or division of both members of an equation by a factor containing the variable and taking the square roots of both members of the equation, are techniques which are fraught with danger and seldom desirable in rectangular representations. They are more frequently acceptable and even desirable in polar representations.

However, we are not suggesting that the teacher should open these questions now. They will be considered in Chapters 5 and 6. To discuss them now would probably only confuse the students. We prefer that the answers to the exercises here be left in the original form obtained without any attempt at simplification. Rather we include this discussion to alert the teacher to the questions laid open and to prepare him or her for the questions that may arise from curious and inquiring students.

Exercises 2-8

1. (a) $-\frac{4}{5}x + \frac{3}{5}y - 3 = 0$

   (b) $\frac{5}{13}x + \frac{12}{13}y - 5 = 0$

   (c) $\frac{3}{\sqrt{13}}x - \frac{2}{\sqrt{13}}y - \frac{6}{\sqrt{13}} = 0$

   (d) $\frac{-5}{\sqrt{34}}x + \frac{3}{\sqrt{34}}y - \frac{12}{\sqrt{34}} = 0$

   (e) $\frac{3}{\sqrt{10}}x - \frac{1}{\sqrt{10}}y - \frac{7}{\sqrt{10}} = 0$

   (f) $\frac{8}{17}x + \frac{15}{17}y - \frac{30}{17} = 0$

   (g) $\frac{12}{13}x - \frac{5}{13}y = 0$

   (h) $y - \frac{20}{7} = 0$

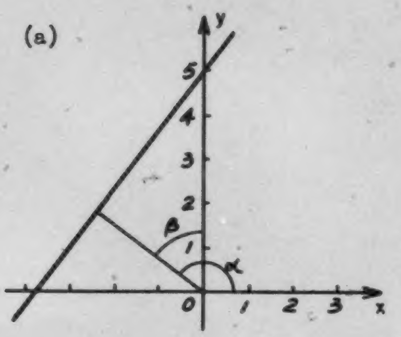
   (i) $-x - \frac{15}{9} = 0$

   (j) $\frac{5x}{13} - \frac{12y}{13} - \frac{60}{13} = 0$

   (k) $-\frac{8}{17}x + \frac{15}{17}y - \frac{120}{17} = 0$

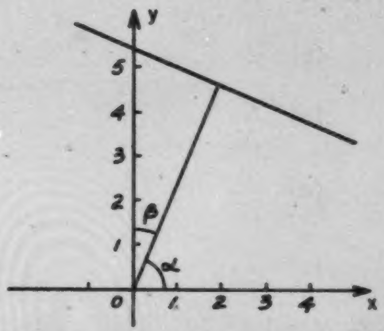
   (l) $\frac{3}{5}x - \frac{4}{5}y - \frac{7}{5} = 0$

2. (a)



$4x - 3y + 15 = 0$

(b)



$5x + 12y - 65 = 0$

Of course this is not an efficient way to draw the graph. The exercise was put in to help familiarize the students with this form of equation for a line.

3. (a) $r \sin \theta = 4$

   (b) $r \cos \theta = 4$

   (c) $\theta = 60^\circ$, or $\theta = \frac{\pi}{3}$

   (d) $r \cos (\theta - 315^\circ) = 3$

   (e) $r \cos (\theta - 300^\circ) = \frac{3}{2}$

   (f) $\theta = 45^\circ$, or $\theta = \frac{\pi}{4}$

   (g) $r \cos (\theta - 150^\circ) = 2$

   (h) $r \cos (\theta + 135^\circ) = 2$

4. (a) $r \cos \theta - 4 = 0$

   (b) $r \sin \theta + 4 = 0$

   (c) $\theta = 90^\circ$, or $\theta = \frac{\pi}{2}$

   (d) $r \cos \theta + r \sin \theta + 2 = 0$

   (e) $3r \cos \theta - 2r \sin \theta + 6 = 0$

   (f) $r \cos \theta + \sqrt{3}\, r \sin \theta - 2 = 0$

   (g) $15r \sin \theta - 8r \cos \theta + 34 = 0$

5. (a) If P_1 is on L, then $|\lambda x_1 + \mu y_1 - p| = 0$. But the distance from P_1 to L is zero when P_1 is on L.

   (b) P_1 is on the same side of L as 0; P_1 is closer than 0 to L. In this case $d(P_1,F) = p - p_1 = |\lambda x_1 + \mu y_1 - p|$.

   (c) P_1 is on the same side of L as 0; P_1 and 0 are equidistant from L. In this case L_1 contains the origin, $p_1 = 0$, and $d(P_1,F) = p - p_1 = |\lambda x_1 + \mu y_1 - p|$.

6. (a) $\frac{58}{13}$

   (b) $\frac{22}{5}$

   (c) $\frac{20}{\sqrt{17}}$

   (d) $\frac{50}{\sqrt{74}}$

   (e) 0

7. A point $P_0 = (x_0,y_0)$ on the bisector if the distance from P to L_1 is equal to the distance from P to L_2.

   Then from our distance formula, we have

$$\left|\frac{3}{5}x - \frac{4}{5}y + 1\right| = \left|\frac{12}{13}x + \frac{5}{13}y - 1\right|$$

   Taking both choices for the signs yields the two desired equations:

$$21x + 77y - 130 = 0$$

   and

$$11[illegible] - 3y = 0$$

8.

$$7x + 9y - 152 = 0$$

   and

$$99x - 77y - 144 = 0$$

9.

$$|\lambda_1 x + \mu_1 y - p_1| = |\lambda_2 x + \mu_2 y - p_2| \quad \text{gives us}$$

$$(\lambda_1 - \lambda_2)x + (\mu_1 - \mu_2)y - (p_1 - p_2) = 0 \quad \text{and}$$

$$(\lambda_1 + \lambda_2)x + (\mu_1 + \mu_2)y - (p_1 + p_2) = 0 .$$

10. $x - 3 = 0$

11. $r \cos \theta - r \sin \theta = 0$

12. $r^2 = 36$

13. $r = 4 \cos \theta$

$r^2 = 4\, r \cos \theta$

$(x^2 + y^2) = 4x$

$x^2 - 4x + y^2 = 0$

When $\theta = \frac{\pi}{2}$, $r = 0$. Thus the pole is in the graph of the original equation. One must make this check because both sides of the equation have been multiplied by r; $r = 0$ is then a root of the new equation.

14. $r = 2a \cos \theta$

Note that the pole is in the graph of the equation. Then $r^2 = 2ar \cos \theta$ or $x^2 + y^2 = 2ax$.

15. (a) $y = \sqrt{3}\, x$

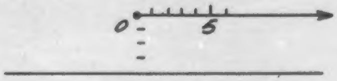
(b) $y + 4 = 0$

(c) $\sqrt{x^2 + y^2} = 5$

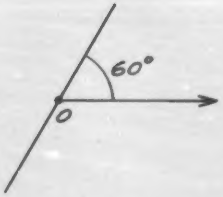
$x^2 + y^2 = 25$

16.

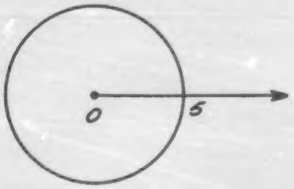
(b)



(a)



(c)



17. (a) $r^2 - 4r \cos \theta = 0$

(b) Note that the pole is in the graph of the equation. Then

$$r^2 = 5r \cos \theta - 3r \sin \theta$$

$$x^2 + y^2 - 5x + 3y = 0$$

(c) $-y = 4$,
or $y = -4$.

(d) $(r^2 + r \sin \theta)^2 = r^2$

Review Exercises - Section 2-6 through Section 2-8

1.

	direction numbers	direction cosines	direction angles (approximately)
(a)	(7,-10)	$(\frac{7}{\sqrt{149}}, -\frac{10}{\sqrt{149}})$	$\alpha = 55^\circ$, $\beta = 145^\circ$
(b)	(25,24)	$(\frac{25}{\sqrt{1201}}, \frac{24}{\sqrt{1201}})$	$\alpha = 44^\circ$, $\beta = 46^\circ$
(c)	(-6,5)	$(\frac{-6}{\sqrt{61}}, \frac{5}{\sqrt{61}})$	$\alpha = 140^\circ$, $\beta = 50^\circ$
(d)	(7,6)	$(\frac{7}{\sqrt{85}}, \frac{6}{\sqrt{85}})$	$\alpha = 41^\circ$, $\beta = 49^\circ$
(e)	(3,-3)	$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$\alpha = 45^\circ$, $\beta = 135^\circ$
(f)	(4,7)	$(\frac{4}{\sqrt{65}}, \frac{7}{\sqrt{65}})$	$\alpha = 60^\circ$, $\beta = 30^\circ$
(g)	(1,2)	$(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$	$\alpha = 63^\circ$, $\beta = 27^\circ$
(h)	(-2,1)	$(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$	$\alpha = 153^\circ$, $\beta = 63^\circ$

2. The points are collinear if two line segments determined by the points have the same slope.

(a) $\frac{13-1}{11-(-4)} = \frac{12}{15} = \frac{4}{5}$

$\frac{13-5}{11-1} = \frac{8}{10} = \frac{4}{5}$ points are collinear

(b) $\frac{-2-7}{1-(-5)} = \frac{-9}{6} = -\frac{3}{2}$

$\frac{-2-(-12)}{1-6} = \frac{10}{-5} = -\frac{2}{1}$ points are not collinear

(c) $\frac{17-(-1)}{23-(-1)} = \frac{18}{24} = \frac{3}{4}$

$\frac{17-(-13)}{23-(-17)} = \frac{30}{40} = \frac{3}{4}$ points are collinear

(d) $\frac{-4-8}{0-(-3)} = \frac{-12}{3} = -4$

$\frac{-4-(-11)}{0-5} = \frac{7}{-5}$ points are not collinear

3. $d(A,B) = \sqrt{41}$ $d(A,C) = \sqrt{53}$ $d(B,C) = 2\sqrt{10}$

4. $\overleftrightarrow{AB} : 4x - 5y + 17 = 0$

$\overleftrightarrow{AC} : 2x + 7y - 1 = 0$

$\overleftrightarrow{BC} : 3x + y - 11 = 0$

5. length of altitude from A : $\frac{19}{\sqrt{10}}$

length of altitude from B : $\frac{38}{\sqrt{53}}$

length of altitude from C : $\frac{38}{\sqrt{41}}$

6. area $(\triangle ABC) = 19$

7. (a) $x(2\sqrt{41} - 4\sqrt{53}) + y(7\sqrt{41} + 5\sqrt{53}) - (\sqrt{41} + 17\sqrt{53}) = 0$

(b) $x(4\sqrt{10} + 3\sqrt{41}) + y(-5\sqrt{10} + \sqrt{41}) + (17\sqrt{10} - 11\sqrt{41}) = 0$

(c) $x(2\sqrt{10} + 3\sqrt{53}) + y(7\sqrt{10} + \sqrt{53}) - (\sqrt{10} + 11\sqrt{53}) = 0$

8. (a) $d(A,L_1) = \frac{3}{\sqrt{13}}$ $d(A,L_2) = \frac{17}{5}$ $d(A,L_3) = \frac{1}{\sqrt{5}}$

(b) $d(B,L_1) = \frac{5}{\sqrt{13}}$ $d(B,L_2) = \frac{14}{5}$ $d(B,L_3) = \frac{4}{\sqrt{5}}$

(c) $d(C,L_1) = \frac{17}{\sqrt{13}}$ $d(C,L_2) = \frac{4}{5}$ $d(C,L_3) = \frac{10}{\sqrt{5}}$

9. (a) $x(10 - 3\sqrt{13}) + y(-15 - 4\sqrt{13}) + (30 + 12\sqrt{13}) = 0$
$x(10 + 3\sqrt{13}) + y(-15 + 4\sqrt{13}) + (30 - 12\sqrt{13}) = 0$

(b) $x(2\sqrt{5} - \sqrt{13}) + y(-3\sqrt{5} + 2\sqrt{13}) + (6\sqrt{5} - 4\sqrt{13}) = 0$
$x(2\sqrt{5} + \sqrt{13}) + y(-3\sqrt{5} - 2\sqrt{13}) + (6\sqrt{5} + 4\sqrt{13}) = 0$

(c) $x(3\sqrt{5} - 5) + y(4\sqrt{5} + 10) + (-12\sqrt{5} - 20) = 0$
$x(3\sqrt{5} + 5) + y(4\sqrt{5} - 10) + (-12\sqrt{5} + 20) = 0$

10. (a) $\frac{6}{\sqrt{13}}$ (b) $\frac{11}{5}$ (c) $\frac{6}{\sqrt{5}}$

11. $P_A(\frac{87}{17}, \frac{92}{17})$ $P_B(-\frac{63}{17}, -\frac{8}{17})$

12. θ_1 82°
θ_2 98°

13. L_1 may be written $3x + 5 - 19 = 0$
L_2 may be written $5x - 3y + 7 = 0$

If $a_1a_2 + b_1b_2 = 0$ the lines are perpendicular

Substituting $(3)(5) + (5)(-3) = 0$
and $L_1 \perp L_2$.

14. Find the angles between L_1 and L_2 , where L_1 contains the points $(3,4)$, $(-1,-1)$: and L_2 contains the points $(-4,6)$, $(3,0)$.

Solution. Since no sense is imposed on L_1 and L_2 we will find their angles of intersection.

We may take as direction numbers for L_1 , $(4,5)$ and for L_2 , $(-7,6)$.

(Why?) Therefore:

$$\text{Cos } \theta = \frac{(4)(-7) + (5)(6)}{\sqrt{4^2 + 5^2}\ \sqrt{(-7)^2 + 6^2}} \approx .034$$

$$\therefore \quad \theta \approx 88^\circ$$

We may, most simply, find the other angle of intersection as the supplement of θ, but it is instructive to use equivalent direction numbers for L_1 which have the effect of reversing the sense induced by the first choice. We use now $(-4,-5)$, and $(-7,6)$ as pairs of direction numbers and get

$$\text{Cos } \theta' = \frac{(-4)(-7) + (-5)(6)}{\sqrt{(-4)^2 + (-5)^2}\ \sqrt{(-7)^2 + 6^2}} \approx -.034$$

$$\therefore \quad \theta' \approx 92^\circ$$

which is, as we expected, supplementary to θ.

15. $$\cos \theta = \frac{\left(-\frac{2}{\sqrt{5}}\right)\left(\frac{7}{\sqrt{58}}\right) + \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{\sqrt{58}}\right)}{\sqrt{\frac{4}{5} + \frac{1}{5}}\ \sqrt{\frac{49}{58} + \frac{9}{58}}} \approx -.647$$

$$\theta \approx 130^\circ$$

16. $A = (3,4)$ $B = (-2,7)$ $C = (6,9)$

$$m_{\overline{AB}} = \frac{7 - 4}{-2 - 3} = -\frac{3}{5}$$

$$m_{\overline{BC}} = \frac{9 - 7}{6 + 2} = \frac{2}{8} = \frac{1}{4}$$

$$m_{\overline{AC}} = \frac{9 - 4}{6 - 3} = \frac{5}{3}$$

Since $m_{AB}\, m_{AC} = (-\frac{3}{5})(\frac{5}{3}) = -1$

$\overline{AB} \perp \overline{AC}$ and $\triangle ABC$ is a right triangle

17. (a) $-\frac{3}{\sqrt{58}}x + \frac{7}{\sqrt{58}}y - \frac{29}{\sqrt{58}} = 0$

(b) $-\frac{20}{29}x + \frac{21}{29}y - 42 = 0$

(c) $\frac{4}{5}x - \frac{3}{5}y - \frac{24}{5} = 0$

(d) $\frac{3}{\sqrt{58}}x - \frac{7}{\sqrt{58}}y = 0$

(e) $x - \frac{7}{5} = 0$

18. (a) $r \cos(\theta - 60) = 1$
(b) $r \cos \theta = -4$
(c) $\theta = 147^\circ$

19. (a) $\sqrt{3}\, x + y = -5$
(b) $3y - 4x = 12$

20. (a) $r\,(8 \cos \theta + 7 \sin \theta) = 56$
(b) $r\,(15 \sin \theta - 8 \cos \theta) = -180$

Challenge Exercises

1. $3x - 4y + c = 0$ or $ax + by + c = 0$, with $\frac{a}{b} = \frac{3}{-4}$.

2. $4x + 3y + c = 0$ or $ax + by + c = 0$, with $\frac{a}{b} = \frac{4}{3}$.

3. $ax + by = 0$

4. $y - 3 = m(x - 2)$

5. $y = \frac{3}{4}(x - 4)$. (Fixing the value of m reduces the family to one member.)

6. $y = -3x + b$ (a pencil of lines.)

7. Let $L_1 : ax + by + c = 0$ and $L_2 : mx + ny + p = 0$ be two intersecting lines. The equations of the lines of the angle bisectors are then

$$x\left(\frac{a}{\sqrt{a^2 + b^2}} - \frac{m}{\sqrt{m^2 + n^2}}\right) + y\left(\frac{b}{\sqrt{a^2 + b^2}} - \frac{n}{\sqrt{m^2 + n^2}}\right) + \left(\frac{c}{\sqrt{a^2 + b^2}} - \frac{p}{\sqrt{m^2 + n^2}}\right) = 0$$

$$x\left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{m}{\sqrt{m^2 + n^2}}\right) + y\left(\frac{b}{\sqrt{a^2 + b^2}} + \frac{n}{\sqrt{m^2 + n^2}}\right) + \left(\frac{c}{\sqrt{a^2 + b^2}} + \frac{p}{\sqrt{m^2 + n^2}}\right) = 0$$

Their slopes are $\dfrac{m\sqrt{a^2+b^2} - a\sqrt{m^2+n^2}}{b\sqrt{m^2+n^2} - n\sqrt{a^2+b^2}}$, $\dfrac{-m\sqrt{a^2+b^2} - a\sqrt{m^2+n^2}}{b\sqrt{m^2+n^2} + n\sqrt{a^2+b^2}}$

The product of the slopes is $\dfrac{-m^2(a^2+b^2)+a^2(m^2+n^2)}{b^2(m^2+n^2)-n^2(a^2 \ b^2)} = \dfrac{-m^2b^2+a^2n^2}{m^2b^2-a^2n^2} = -1$

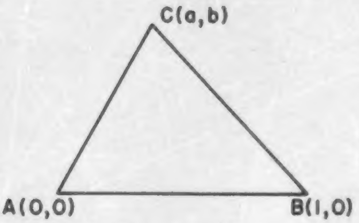
Hence, the lines of the bisectors are perpendicular.

8. $L = \{(x,y):\ ax + by + c = f(x,y) = 0\}$ and

$L_1 = \{(x,y):\ ax_1 + by_1 + c = f(x_1,y_1) = 0\}$

The direction numbers of each line are (a,b) . Therefore the lines are parallel.

9. Given $\triangle ABC$ with vertices $A(0,0)$, $B(1,0)$ and $C(a,b)$.
To prove that the altitudes are congruent at a point H and find the coordinates of H .



the slope of $\overline{AB}$ is 0

the slope of $\overline{AC}$ is $\frac{b}{a}$

the slope of $\overline{BC}$ is $\frac{b}{a-1}$

The slope of the altitude from A is $-\frac{a-1}{b}$

The slope of the altitude from B is $-\frac{a}{b}$

The altitude from A is represented by $y = -\frac{a-1}{b}x$

The altitude from B is represented by $y = -\frac{a}{b}(x-1)$

If the altitudes are concurrent, $-\frac{a-1}{b}x = -\frac{a}{b}(x-1)$

and $x = a$ and $y = -\frac{a(a-1)}{b}$

the equation of the altitude from C is $x = a$ and the point of intersection of the other two altitudes is clearly on this line.

10. The midpoint of $\overline{AB} = (\frac{1}{2}, 0)$

The midpoint of $\overline{BC} = (\frac{a+1}{2}, \frac{b}{2})$

The midpoint of $\overline{AC} = (\frac{a}{2}, \frac{b}{2})$

The median from A is represented by

$$y = (\frac{b}{a} + 1)x$$

The median from B is represented by

$$y = \frac{b}{a-2}(x-1)$$

These two medians intersect at the point

$$(\frac{a+1}{3}, \frac{b}{3})$$

The median from C is represented by

$$y = \frac{b}{a - \frac{1}{2}}(x - \frac{1}{2})$$

and the point $(\frac{a+1}{3}, \frac{b}{3})$ is contained in this line.

Therefore the medians are concurrent at $(\frac{a+1}{3}, \frac{b}{3})$

11. The bisector of $\angle A$ is given by

$$y = \frac{bx - ay}{\sqrt{a^2 + b^2}}$$ and solving for y,

$$y = \frac{bx}{\sqrt{a^2 + b^2} + a} \qquad (1)$$

The bisector of $\angle B$ is given by

$$y = \frac{b - bx - (1-a)y}{\sqrt{b^2 + (1-a)^2}}$$ and solving for y,

$$y = \frac{b(1-x)}{\sqrt{b^2 + (1-a)^2} - a + 1} \qquad (2)$$

Equating (1) and (2)

$$\frac{bx}{\sqrt{a^2 + b^2} + a} = \frac{b(1 - x)}{\sqrt{b^2 + (1 - a)^2} + 1 - a}$$

Solving for x we get,

$$x = \frac{\sqrt{a^2 + b^2} + a}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}}$$

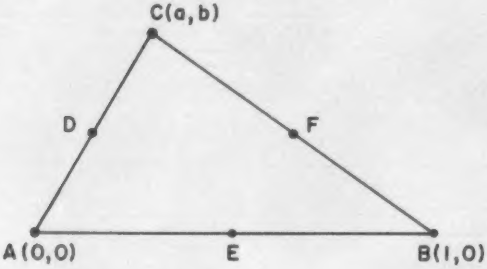
Substituting x into equation (1) ,

$$y = \frac{b}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}}$$

So the point of intersection is

$$\frac{\sqrt{a^2 + b^2} + a}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}} , \frac{b}{\sqrt{b^2 + (1 - a)^2} + 1 + \sqrt{a^2 + b^2}}$$

12.



Midpoint of $\overline{AC} = (\frac{a}{2}, \frac{b}{2}) = D$

Midpoint of $\overline{BC} = (\frac{a + 1}{2}, \frac{b}{2}) = F$

Midpoint of $\overline{AB} = (\frac{1}{2}, 0) = E$

Slope of $\overline{AB} = 0$

Slope of $\overline{AC} = \frac{b}{a}$

Slope of $\overline{BC} = \frac{b}{a - 1}$

Equations of perpendicular bisector through D =

$$y = -\frac{a}{b}x + \frac{a^2}{2b} + \frac{b}{2} \qquad (1)$$

Equation of perpendicular bisector through E =

$$x = \frac{1}{2} \qquad (2)$$

Equation of perpendicular bisector through F =

$$y = -\frac{a-1}{b}x + \frac{a^2-1}{2b} + \frac{b}{2} \qquad (3)$$

If $x = \frac{1}{2}$ is substituted into equation (1) and (3) the values of y are the same. Therefore the perpendicular bisectors are concurrent at

$$\left(\frac{1}{2}, \frac{a^2-a}{2b} + \frac{b}{2}\right)$$

13.

$$H = \left(a, \frac{a(1-a)}{b}\right)$$

$$G = \left(\frac{a+1}{3}, \frac{b}{3}\right)$$

$$E = \left(\frac{1}{2}, \frac{a^2+b^2-a}{2b}\right)$$

The slope of $\overline{HG} = \dfrac{\frac{a-a^2}{b} - \frac{b}{3}}{a - \frac{a+1}{3}} = \dfrac{3a-3a^2-b^2}{(2a-1)b}$;

The slope of $\overline{HE} = \dfrac{\frac{a-a^2}{b} - \frac{a^2+b^2-a}{2b}}{a - \frac{1}{2}} = \dfrac{3a-3a^2-b^2}{(2a-1)b}$.

Therefore, the points are collinear. An equation of the line is

$$(3a^2 + b^2 - 3a)x + (2ab - b)y + a - a^3 - ab^2 = 0 .$$

Illustrative Test Items - Sections 2-6 through 2-8

1. Find a pair of direction numbers for the line $\overleftrightarrow{PQ}$.

   (a) $P = (2,3)$, $Q = (4,5)$.
   (b) $P = (1,-4)$, $Q = (7,4)$.
   (c) $P = (-2,7)$, $Q = (4,3)$.
   (d) $P = (-2,-3)$, $m = -1$.
   (e) $P = (-1,7)$, $\alpha = 150^\circ$
   (f) x-intercept 4 ; y-intercept 3 .

2. Find a pair of direction cosines for a line,

   (a) $L = \{(x,y): x - y + 2 = 0\}$.
   (b) containing $(3,5)$ and $(1,7)$.
   (c) with slope $-\sqrt{3}$.
   (d) with inclination $\alpha = 30^\circ$.
   (e) parallel to the x-axis .
   (f) perpendicular to the x-axis .

3. Find direction angles for

   (a) the line containing $(-1,-3)$ and $(-3,-1)$.

   (b) the ray emanating from the origin and containing the point $(6,-6\sqrt{3})$.

   (c) the line with equation $\sqrt{3}\,x + y - 7 = 0$.

   (d) the normal segment to $L = \{(x,y): x + \sqrt{3}\,y + 7 = 0\}$.

4. Which, if any, of the lines with the given equations are parallel? perpendicular? the same line?

   L_1: $y - 1 = \frac{2}{3}(x + 2)$
   L_4: $y = \frac{2}{3}x - \frac{1}{3}$
   L_2: $\frac{x}{4} + \frac{y}{6} = 1$
   L_5: $\frac{x + 2}{1 + 2} = \frac{y - 1}{3 - 1}$
   L_3: $3x + 2y + 3 = 0$

5. Find the cosine of the least angle between the pairs of lines with the indicated equations.

   (a) $x + 3y - 1 = 0$; $2x + 3y - 7 = 0$.
   (b) $2x + 4y - 5 = 0$; $3x + 4y - 1 = 0$.
   (c) $x - y + 13 = 0$; $5x + 3y + 12 = 0$.

6. Let $L = \{(x,y): 4x - 7y + 13 = 0\}$. Write an equation in general form of a line

(a) parallel to L and containing the point $(3,2)$.

(b) perpendicular to L and containing the origin.

(c) parallel to L and with x-intercept 4 .

(d) perpendicular to L and containing the point $(3,2)$.

7. Find an equation of the perpendicular bisector of $\overline{AB}$, where $A = (1,-3)$, $B = (7,1)$.

8. Let $A = (1,1)$, $B = (8,3)$, and $C = (5,8)$. Find the area of triangle ABC .

9. A line L_1 makes an angle whose cosine is $\frac{2}{5}\sqrt{5}$ with $L_2 = \{(x,y): 2x + y - 7 = 0\}$. What is the slope of L_1 ? Find an equation of L_1 if it contains the point $(-4,2)$.

10. Find the normal form of each of the following equations.

(a) $3x - 4y + 15 = 0$

(b) $\frac{x - 2}{5 - 2} = \frac{y + 1}{2 + 1}$

(c) $y - 7 = \frac{7}{3}(x + 4)$

(d) $\frac{x}{5} + \frac{y}{12} = 1$

(e) $y = \frac{8}{15}x - 2$

(f) $\frac{x + 3}{21 + 3} = \frac{y - 4}{11 - 4}$

(g) $7x - 2y = 0$

(h) $7 - 3y = 0$

11. Find the distance between P and L :

(a) $P = (5,10)$; $L = \{(x,y): 3x - 4y + 10 = 0\}$.

(b) $P = (5,-1)$; $L = \{(x,y): 12x - 5y + 26 = 0\}$.

(c) $P = (6,4)$; $L = \{(x,y): x + 2y - 4 = 0\}$.

(d) $P = (7,-3)$; $L = \{(x,y): 2x - 3y + 5 = 0\}$.

12. Find equations of the lines bisecting the angles formed by

    (a) $L_1 = \{(x,y):\ 3x - 4y + 5 = 0\}$ and $L_2 = \{(x,y):\ 5x - 12y + 26 = 0\}$

    (b) $L_1 = \{(x,y):\ x + y - 1 = 0\}$ and $L_2 = \{(x,y):\ 8x - 15y + 34 = 0\}$.

13. Write in polar form the equations of the following lines:

    (a) parallel to the polar axis and 2 units above it.

    (b) perpendicular to the polar axis and 3 units to the right of the pole.

    (c) containing the point $(-2, \frac{5\pi}{4})$ and having inclination $\frac{3\pi}{4}$.

    (d) through the pole with slope -1 .

14. Transform each of the following equations into polar coordinates.

    (a) $3x - 2y + 5 = 0$

    (b) $7x + 8y - 56 = 0$

    (c) $x^2 + y^2 = 25$

    (d) $y = x^2 + 4x + 4$

15. Transform each of the following equations into rectangular coordinates.

    (a) $r \cos \theta = 4$

    (b) $2r \cos \theta + 5r \sin \theta = 6$

    (c) $r = 3 \sin \theta$

    (d) $r \cos (\theta - \frac{\pi}{2}) = 4$

16. Let the vertices of the triangle ABC be $A = (-4,2)$, $B = (6,6)$, $C = (4,-4)$.

    (a) Find the lengths of the sides.

    (b) Find the equations of the lines containing the sides.

    (c) Find an equation of the perpendicular bisector of side $\overline{AC}$.

    (d) Find an equation of the line containing the altitude to side $\overline{AC}$.

    (e) Find the length of the altitude to side $\overline{AC}$.

    (f) Find an equation of the line containing the median to side $\overline{AC}$.

    (g) Find the length of the median to side $\overline{AC}$.

    (h) Find the area of the triangle.

    (i) Find the centroid of triangle ABC (intersection of the medians).

    (j) Find an equation of the line containing the bisector of $\angle A$.

Answers

1. (a) $(2,2)$, or equivalent pair
   (b) $(6,8)$, or equivalent pair
   (c) $(6,-4)$, or equivalent pair
   (d) $(1,-1)$, or equivalent pair.
   (e) $(-\sqrt{3},1)$, or equivalent pair.
   (f) $(-4,3)$, or equivalent pair.

2. (a) $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, or $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$
   (b) $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, or $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$
   (c) $(\frac{1}{2},-\frac{\sqrt{3}}{2})$, or $(-\frac{1}{2},\frac{\sqrt{3}}{2})$
   (d) $(\frac{\sqrt{3}}{2},\frac{1}{2})$, or $(-\frac{\sqrt{3}}{2},-\frac{1}{2})$
   (e) $(1,0)$, or $(-1,0)$
   (f) $(0,1)$, or $(0,-1)$

3. (a) $\alpha = 135^\circ$, $\beta = 45^\circ$; or $\alpha = 45^\circ$, $\beta\ 135^\circ$.
   (b) $\alpha = 60^\circ$, $\beta = 150^\circ$.
   (c) $\alpha = 120^\circ$, $\beta = 30^\circ$; or $\alpha = 60^\circ$, $\beta = 150^\circ$.
   (d) $\alpha = 120^\circ$, $\beta = 150^\circ$.

4. L_1 and L_5 are the same.

   L_1 , L_4 , and L_5 are parallel

   L_2 and L_3 are parallel

   L_1 , L_4 , and L_5 are perpendicular to L_2 and L_3 .

5. (a) $\frac{11}{\sqrt{130}}$ (b) $\frac{11}{5\sqrt{5}}$ (c) $\frac{1}{\sqrt{17}}$

6. (a) $4x - 7y + 2 = 0$
   (b) $7x + 4y = 0$
   (c) $4x - 7y - 16 = 0$
   (d) $7x + 4y - 29 = 0$

7. $3x + 2y - 10 = 0$

8. $20\frac{1}{2}$

9. $m = -\frac{3}{4}$

   $3x + 4y + 4 = 0$

10. (a) $-\frac{3}{5}x + \frac{4}{5}y - 3 = 0$

(b) $\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - \frac{3}{\sqrt{2}} = 0$

(c) $\frac{-7}{\sqrt{58}}x + \frac{3}{\sqrt{58}}y - \frac{49}{\sqrt{58}} = 0$

(d) $\frac{12}{13}x + \frac{5}{13}y - \frac{60}{13} = 0$

(e) $\frac{8}{17}x - \frac{15}{17}y - \frac{30}{17} = 0$

(f) $-\frac{7}{25}x + \frac{24}{25}y - \frac{117}{25} = 0$

(g) $\frac{7}{\sqrt{53}}x - \frac{2}{\sqrt{53}}y = 0$

(h) $y - \frac{7}{3} = 0$

11. (a) 3
(b) 7
(c) $2\sqrt{5}$
(d) $\frac{28}{\sqrt{13}}$

12. (a) $14x + 8y - 65 = 0$ and $64x - 112y + 195 = 0$

(b) $x(17 - 8\sqrt{2}) + y(17 + 15\sqrt{2}) - (17 + 34\sqrt{2}) = 0$ and

$x(17 + 8\sqrt{2}) + y(17 - 15\sqrt{2}) - (17 - 34\sqrt{2}) = 0$.

13. (a) $r \cos (\theta - \frac{\pi}{2}) = 2$

(b) $r \cos \theta = 3$

(c) $r \cos (\theta - \frac{\pi}{4}) = 2$

(d) $\theta = \frac{3\pi}{4}$

14. (a) $3r \cos \theta - 2r \sin \theta + 5 = 0$
(b) $7r \cos \theta + 8r \sin \theta - 56 = 0$

(c) $r^2 = 25$

(d) $r \sin \theta = r^2 \cos^2 \theta + 4r \cos \theta + 4 = (r \cos \theta + 2)^2$

15. (a) $x = 4$
(b) $2x + 5y = 6$
(c) $x^2 + y^2 = 3y$
(d) $y = 4$

16. (a) $d(A,B) = 2\sqrt{29}$; $d(B,C) = 2\sqrt{26}$; $d(A,C) = 10$.
(b) $\overleftrightarrow{AB}$: $2x - 5y + 18 = 0$

$\overleftrightarrow{BC}$: $5x - y - 24 = 0$

$\overleftrightarrow{AC}$: $3x + 4y + 4 = 0$

(c) $4x - 3y - 3 = 0$
(d) $4x - 3y - 6 = 0$
(e) $\frac{46}{5} = 9.2$
(f) $7x - 6y - 6 = 0$
(g) $\sqrt{85}$
(h) 46
(i) $(2,\frac{4}{3})$
(j) $x(3\sqrt{29} - 10) + y(4\sqrt{29} + 25) + (4\sqrt{29} - 90) = 0$

Chapter 3

VECTORS AND THEIR APPLICATION

3-1. *Why Study "Vectors"?*

91 In the opening paragraphs reference is made to the increasing importance of vectors and vector methods in the fields of applied mathematics, science, and engineering. You need only pick up any text in these subjects to be assured of the accuracy of this statement. Most recent books in calculus (e.g., *Calculus and Analytic Geometry* by G.B. Thomas) make considerable use of vector methods. You may like to read *Analytic Geometry: A Vector Approach* by Charles Wexler for an extensive treatment of this subject.

It is quite likely that most of your students will go on to study calculus and more advanced mathematics. Most students in science and engineering are now encouraged to take courses in vector analysis and linear algebra. The latter course starts with vector algebra and uses it to approach the subject of matrices. In this context, a vector is a row or column of a matrix. Our approach is from the geometric point of view (as is vector analysis) but the two are clearly closely related.

The beginnings of this subject can be found in the writings of Aristotle, and later in the works of Galileo (1564-1642, Italian). However, serious study of the subject began with William Rowan Hamilton (1805-1865, Irish) and Herman Grassmann (1809-1877, German). Their work was dependent upon the earlier development of analytic geometry. Hamilton was inspired by problems arising from Newtonian physics and astronomy. In solving problems related to the motion of particles, Hamilton needed a non-commutative algebra. The quaternion $A = a_0 + a_1 i + a_2 j + a_3 k$ (where $i^2 = j^2 = k^2 = ijk = -1$ and the a's are real), provided the answer since, for example, $i \cdot j = -j \cdot i$. The quaternion led to the vector and, in the cross-product of vectors, $A \times B = -B \times A$. (See this Commentary on Section 3-7).

Grassmann approached the subject of vectors from the algebraic point of view. He was seeking an algebraic method of extending geometry from three into n dimensions. A vector in two dimensions is defined as an ordered pair of real numbers and in three dimensions as an ordered triple of real numbers. In n dimensions, a vector is an ordered n-tuple of real numbers. This is the approach used today in the study of vector spaces in modern algebra.

If your students have already studied vectors in SMSG "Geometry with Coordinates", "Intermediate Mathematics", or "Matrix Algebra", a large part of the material in this chapter will serve as a review. Some time should be spent, however, in analyzing the different approaches to the subject. In this way the students will review the topic from another point of view. Some of the subject matter and many of the problems are new to all.

3-2. Directed Line Segments and Vectors.

92 For more information regarding directed line segments, you should read the SMSG "Intermediate Mathematics", p. 629-634.

Probably the most distinctive part of our approach to the study of vectors lies in our definition of a vector. Since there is no way to distinguish any directed line segment from another with the same magnitude and sense of direction, it is therefore reasonable to define a vector as an infinite set of equivalent directed line segments. Any member of the set can be used to represent this vector. The origin-vector (a new term created here) is very often used to represent the set because of its convenience in geometric proofs and in the study of vector components.

Unless specific geometric conditions obtain, our approach to the subject also gives us the freedom to use free vectors or bound vectors as we choose. The "Origin Principle" on page 93 and the "Origin-Vector Principle" on page 96 are carefully and explicitly stated to make this point clear.

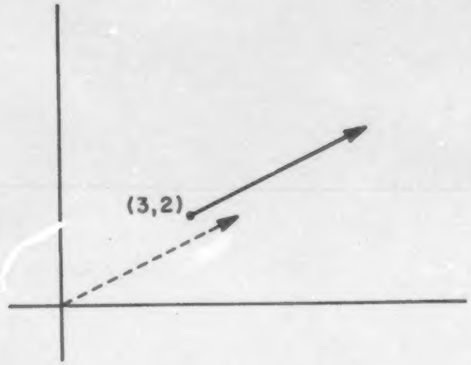
93 The question of equality or inequality of vectors refers only to sets. When we say "two vectors are equal" we are only talking about the same infinite set of directed line segments. Thus "equality" really means "identity". The use of the term in this sense is consistent with its use in all other SMSG texts. For example in earlier texts, if $\overline{AB} = \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are identically the same segment, with $A = C$ and $B = D$.

However, in applications of vectors, it is convenient to use the term vector, as we state in the text, to mean a single member of the set. We consider it proper to do this when there is no danger of ambiguity. The students will then be on more familiar ground when they meet vectors in other courses.

96 The discussion surrounding the origin-vector principle is of greatest importance. You will have many occasions to refer to it in the succeeding sections, particularly in Chapter 4, where many proofs of geometric theorems are discussed.

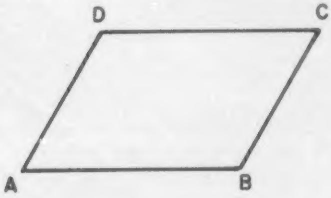
Exercises 3-2

1.



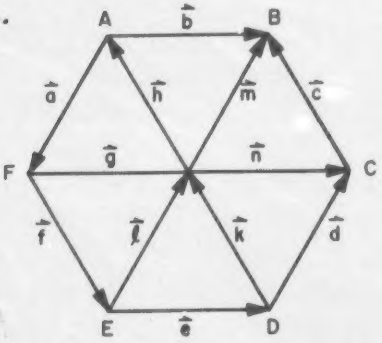
2. $\overrightarrow{FE}$ and $\overrightarrow{JI}$; $\overrightarrow{LK}$ and $\overrightarrow{UT}$; $\overrightarrow{QR}$, $\overrightarrow{OP}$ and $\overrightarrow{MN}$, $\overrightarrow{QS}$ and $\overrightarrow{TV}$
Each set is a representation of the same vector.

3.



$\overrightarrow{DC}$ and $\overrightarrow{AB}$
$\overrightarrow{CD}$ and $\overrightarrow{BA}$
$\overrightarrow{AD}$ and $\overrightarrow{BC}$
$\overrightarrow{DA}$ and $\overrightarrow{CB}$

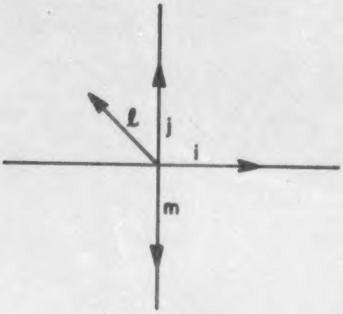
4.

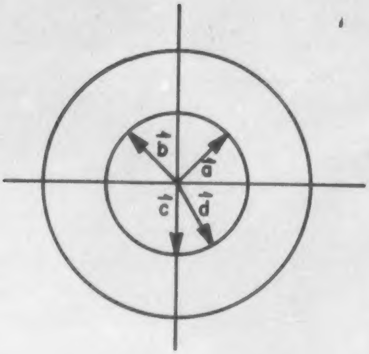


(a) $\vec{\ell} = \vec{m}$
$\vec{e} = \vec{b}$
$\vec{\ell} = \vec{d}$
$\vec{h} = \vec{k}$
(and others)

(b) $\vec{a} = -\vec{\ell}$
$\vec{f} = -\vec{c}$
$\vec{g} = -\vec{n}$
$\vec{h} = -\vec{f}$
(and others)

5.





6. Motion of a car, winds, weight, momentum, angular momentum, electrical and magnetic fields, etc.

3-3. Sum and Difference of Vectors. Scalar Multiplication.

97 The definition presented on this page is concerned only with the sum of two non-zero vectors not lying in the same line.

If $\vec{A}$ and $\vec{B}$ lie in the same line and have the same sense of direction, then $\vec{A} + \vec{B}$ is a vector in the same line with the same sense of direction and with magnitude $|\vec{A}| + |\vec{B}|$. If $\vec{A}$ and $\vec{B}$ have different senses of direction and, let us say, $|\vec{A}| > |\vec{B}|$, then $\vec{A} + \vec{B}$ will have the direction of $\vec{A}$ and magnitude $|\vec{A}| - |\vec{B}|$.

98 By part (2) of the definition of the sum of two vectors, $\vec{P} + \vec{P}$ is a vector with magnitude twice the magnitude of $\vec{P}$. Similarly $(\vec{P} + \vec{P}) + \vec{P}$ is a vector with magnitude 3 times the magnitude of $\vec{P}$. Thus the definition of $r\vec{P}$ generalizes naturally from what we think $2\vec{P}$ and $3\vec{P}$ should be (neither being defined at this point).

99 An emphasis on subtraction of vectors defined in terms of addition should be made. This should be done not only for purely algebraic reasons, but also to simplify finding the difference of two vectors in a vector diagram.

Exercises 3-3

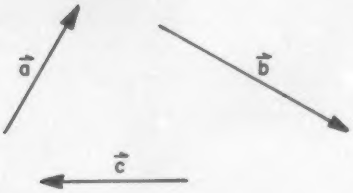
1. (a) $\vec{C}$
   (b) $\vec{C}$
   (c) $\vec{E}$ (requires assumptions that vector addition is associative and that diagonals of a parallelogram bisect each other)
   (d) 2 (requires second assumption in part c)
   (e) $\vec{C}$

2. (a) $\vec{e} = -\vec{a}$
   $\vec{d} = -\vec{b}$
   $\vec{e} = \vec{a} + \vec{b}$

   (b) (i) $\vec{e} = \vec{a} - \vec{d}$
   (ii) $\vec{e} = \vec{a} + \vec{b}$
   (iii) $\vec{e} = \vec{b} - \vec{c}$
   (iv) $\vec{e} = -\vec{c} - \vec{d}$

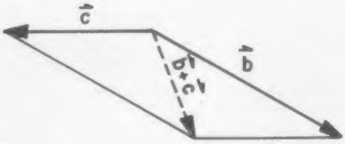
   (c) (i) $\vec{0}$
   (ii) $\vec{0}$

3.



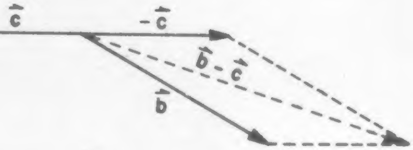
$\vec{a}$
$\vec{b}$
$\vec{c}$

(a) $\vec{b} + \vec{c}$



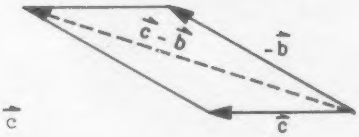
$\vec{c}$
$\vec{b}$
$\vec{b}+\vec{c}$

(b) $\vec{b} - \vec{c}$



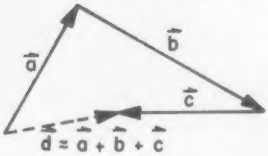
$\vec{c}$
$-\vec{c}$
$\vec{b}-\vec{c}$
$\vec{b}$

(c) $\vec{c} - \vec{b}$



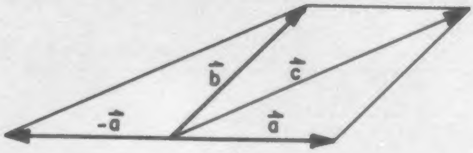
$\vec{c}-\vec{b}$
$-\vec{b}$
$\vec{c}$

(d) $\vec{a} + \vec{b} + \vec{c}$



$\vec{a}$
$\vec{b}$
$\vec{c}$
$\vec{d} = \vec{a} + \vec{b} + \vec{c}$

4.



$\vec{a} + \vec{b} = \vec{c}$

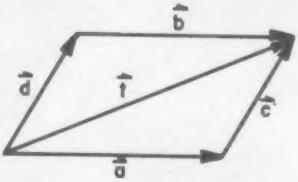
It can also be seen that $-\vec{a} + \vec{c} = \vec{b}$ $\therefore \vec{b} = \vec{c} - \vec{a}$

5. (a) $\frac{1}{2}$ (d) $\frac{3}{4}$

(b) 2 (e) $\frac{3}{2}$

(c) -1 (f) $-\frac{3}{5}$

6.



From the diagram above $\vec{a} = \vec{b}$ and $\vec{c} = \vec{d}$

also $\vec{b} + \vec{d} = \vec{t}$ and $\vec{a} + \vec{c} = \vec{t}$.

$$\therefore \vec{a} + \vec{c} = \vec{b} + \vec{d}$$

7. $|4\vec{A}| = 12$

$|-5\vec{A}| = 15$

$-|5\vec{A}| = -15$

8. Since $\vec{a} = \vec{b}$, $\vec{a}$ and $\vec{b}$ are representatives of the same infinite set of equivalent directed line segments. Thus

$$|\vec{a}| = |\vec{b}| \text{ and } \vec{a} \parallel \vec{b}$$

Now $r\vec{a} \parallel \vec{a}$ and $r\vec{a}$ is r times as large as $\vec{a}$. Also $r\vec{b} \parallel \vec{b}$ and is r times as large as $\vec{b}$. Thus

$$|r\vec{a}| = |r\vec{b}| \text{ and } r\vec{a} \parallel r\vec{b}$$

$$\therefore r\vec{a} = r\vec{b}$$

9. $|k\vec{b}|$ is equal to the magnitude of $\vec{a}$.

10. (a) $\overrightarrow{b} = \overrightarrow{e}$

$\overrightarrow{h} = \overrightarrow{p}$

$\overrightarrow{g} = -\overrightarrow{n}$

$\overrightarrow{d} = -\overrightarrow{a}$

$\overrightarrow{c} = -\overrightarrow{f}$

$\overrightarrow{\ell} = \overrightarrow{m}$

(b) $\overrightarrow{\ell} - \overrightarrow{k} = \overrightarrow{b}$

$\overrightarrow{g} + \overrightarrow{b} = \overrightarrow{a} + \overrightarrow{m}$

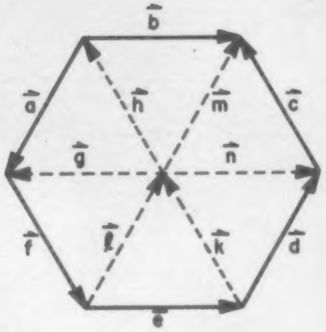
$\overrightarrow{\ell} - \overrightarrow{c} + \overrightarrow{g} = \overrightarrow{0}$

$\overrightarrow{h} + \overrightarrow{n} = \overrightarrow{\ell} + \overrightarrow{k}$

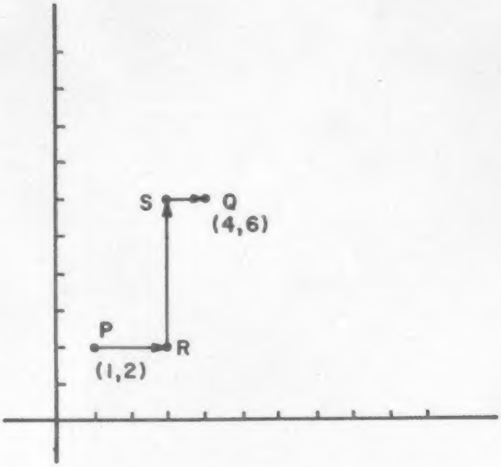
$\overrightarrow{b} + \overrightarrow{e} = 2\overrightarrow{n}$

$\overrightarrow{e} + \overrightarrow{h} = -\overrightarrow{a}$

(and others)


$\overrightarrow{b}$
$\overrightarrow{a}$
$\overrightarrow{h}$
$\overrightarrow{m}$
$\overrightarrow{c}$
$\overrightarrow{g}$
$\overrightarrow{n}$
$\overrightarrow{f}$
$\overrightarrow{\ell}$
$\overrightarrow{k}$
$\overrightarrow{d}$
$\overrightarrow{e}$

11.

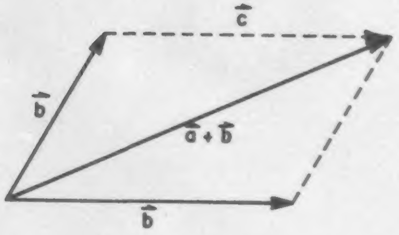


One example: One could follow the path from P to R, from R to S, from S to Q.

12. (a) not necessarily

(b) yes

13.

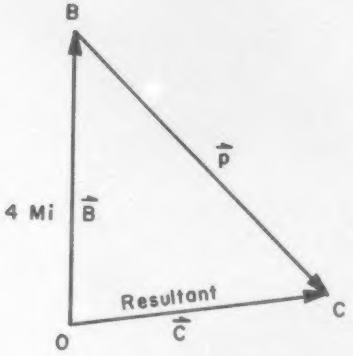

$\vec{c}$
$\vec{b}$
$\vec{a} + \vec{b}$
$\vec{b}$

$|\vec{a}|$ is length of $\vec{a}$
$|\vec{b}|$ is length of $\vec{b}$
$|\vec{a} + \vec{b}|$ is length of $\vec{a} + \vec{b}$

Since $\vec{c}$ is equivalent to $\vec{a}$, then $|\vec{c}| = |\vec{a}|$.
Since the sum of the lengths of two sides of a triangle is greater than or equal that of the third, we have

$$|\vec{c}| + |\vec{b}| \geq |\vec{a} + \vec{b}| .$$
$$\therefore\ |\vec{a}| + |\vec{b}| \geq |\vec{a} + \vec{b}| .$$

14.


B
$\vec{p}$
4 Mi
$\vec{B}$
Resultant
$\vec{C}$
O
C

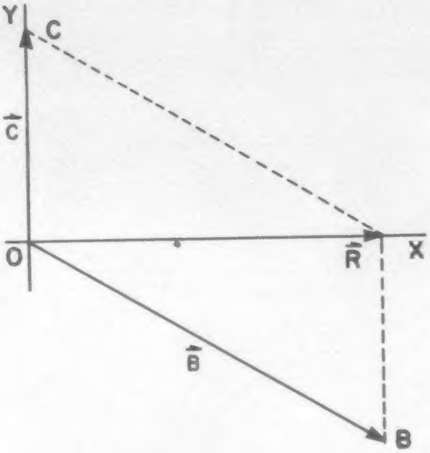
$|\vec{B}| = 2''$

$|\vec{p}| = 2\frac{1}{2}''$

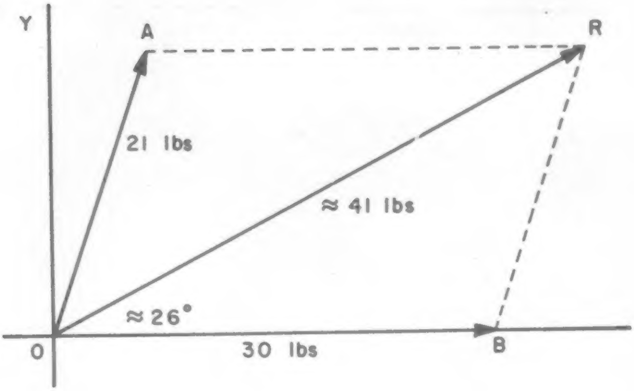
$\vec{C}$ is the resultant

$|\vec{C}| \approx 1\frac{3}{4}''$, representing approximately $3\frac{1}{2}$ miles in the direction indicated.

15. Let the speed and direction of the current be represented by $\vec{C}$ along the y-axis. Let the actual speed and direction of the boat be represented by $\vec{R}$. We want to find the vector $\vec{B}$ representing the boat's motion in still water which when added to $\vec{C}$ represents the combined effect of current and engine on the boat. $\vec{R} = \vec{C} + \vec{B}$. $|\vec{B}|$ represents 6 m.p.h. at $\angle ROB$.



16.



17. $\vec{A}$ and $\vec{B}$ are distinct vectors

Let A have coordinates (a,b), B coordinates (c,d)

Then $\vec{-B}$ has its terminal point at (-c,-d)

and $\vec{-A}$ has its terminal point at (-a,-b).

Thus $\vec{A} - \vec{B}$ has its terminal point at (a - c, b - d)

and $\vec{B} - \vec{A}$ has its terminal point at (c - a, d - b).

Case one: $b \neq d$

Then slope of line $\overleftrightarrow{AB}$ is given by $\frac{b-d}{a-c}$

and slope of line $\overleftrightarrow{OC}$ is given by $\frac{(b-d)-0}{(a-c)-0} = \frac{b-d}{a-c}$.

Therefore the lines are parallel.

Case two: $b = d$.

Then line $\overleftrightarrow{AB}$ has no slope defined, but it is parallel to the line $x = 0$, which is the line OC.

The proof that $\vec{B} - \vec{A}$ lies on a line parallel to the line through A and B is similar.

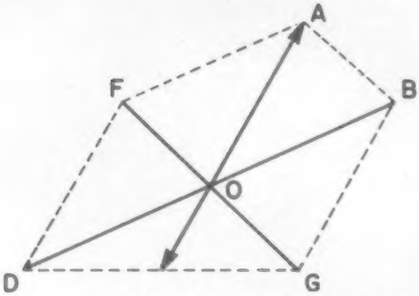
If $b \neq d$ then $m(\overleftrightarrow{AB}) = \frac{u - b}{c - a}$

and $m(\overleftrightarrow{OD}) = \frac{(d - b) - 0}{(c - a) - 0} = \frac{d - b}{c - a}$

So the lines are parallel.

If $b = d$, then $\overleftrightarrow{AB}$ is parallel to the line $x = 0$ which is $\overleftrightarrow{OD}$.

Alternatively, we need not use coordinates:



Let $\vec{D} = -\vec{B}$ and $\vec{E} = -\vec{A}$. $\vec{A} - \vec{B}$ is the vector determined by the vector opposite O in the parallelogram formed with $\overline{OA}$ and $\overline{OD}$ as sides. Hence $\vec{F} = \vec{A} - \vec{B}$. But $d(F,A) = d(D,O)$ and $d(D,O) = d(O,B)$. So $d(F,A) = d(O,B)$. Because $\overleftrightarrow{OD} = \overleftrightarrow{OB}$ and $\overleftrightarrow{FA} \parallel \overleftrightarrow{OD}$, we see that $\angle FAO = \angle BOA$. With $d(O,A) = d(A,O)$ we now know that $\triangle FAO \cong \triangle BOA$.

We get $d(F,O) = d(a,B)$ which tells us that OFAB is a parallelogram since we already have $d(F,A) = d(O,B)$. So $\vec{F} = \vec{A} - \vec{B}$ lies on a line parallel to $\overleftrightarrow{AB}$

18. Given that $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$, are consecutive vector sides of a quadrilateral. We wish to prove that the figure is a parallelogram if and only if $\vec{b} + \vec{d} = \vec{0}$. We must show that:

(1) if $\vec{b} + \vec{d} = \vec{0}$, then the quadrilateral is a parallelogram and that
(2) if the quadrilateral is a parallelogram, then $\vec{b} + \vec{d} = \vec{0}$.

Proof:

(1) Assume $\vec{b} + \vec{d} = \vec{0}$

$\vec{b} = -\vec{d}$

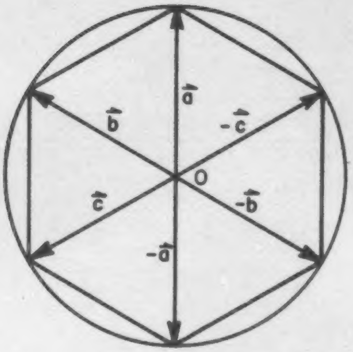
$\therefore$ $\vec{b}$ and $\vec{d}$ are parallel, have the same magnitude and are opposite sides.

$\therefore$ Quadrilateral is a parallelogram.

(2) Assume the quadrilateral is a parallelogram. Then the opposite sides must be equal and parallel; i.e., $\vec{b} = -\vec{d}$.

$\therefore$ $\vec{b} + \vec{d} = \vec{0}$.

19.

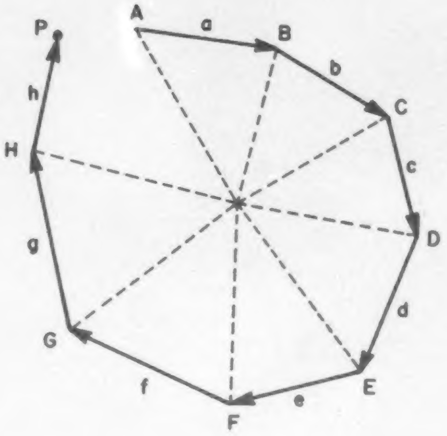


The diagram above shows labeling which leads to a simple proof.

To prove: The sum of six vectors drawn from the center of a regular hexagon to its vertices is zero.

$$\vec{a} + (-\vec{a}) + \vec{b} + (-\vec{b}) + \vec{c} + (-\vec{c}) = \vec{0} .$$

20.



(1) Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, $\overrightarrow{CD} = \vec{c}$, ... , $\overrightarrow{PA} = \vec{p}$.

(2) Note that for triangle ABO , we have

$$\overrightarrow{AB} + \overrightarrow{BO} = - \overrightarrow{OA}$$
$$\therefore \quad \overrightarrow{AB} + \overrightarrow{BO} + \overrightarrow{OA} = \vec{0}$$

(3) Then if we divide our polygon into triangles as shown, we have:

$$(\overrightarrow{AB} + \overrightarrow{BO} + \overrightarrow{OA}) + (\overrightarrow{BC} + \overrightarrow{CO} + \overrightarrow{OB}) + \dots + \overrightarrow{AO} = \vec{0} .$$

But $\overrightarrow{AO} = -\overrightarrow{OA}$, $\overrightarrow{BD} = -\overrightarrow{DB}$, etc. ...

$\therefore$ (4) $\vec{a} + \vec{b} + \vec{c} + \dots + \vec{p} = \vec{0}$, or $\overrightarrow{AB} + \overrightarrow{BC} + \dots \overrightarrow{PA} = \vec{0}$.

3-4. Properties of Vector Operations.

104 The purpose of this section is to develop some algebraic structure for the operations of vector addition and scaler multiplication.

Perhaps the best way of showing the associative property by means of Figure 3-9 is to consider the quadrilateral whose vertices are the terminal points of $\overrightarrow{Q}$, $\overrightarrow{P} + \overrightarrow{Q}$, $\overrightarrow{Q} + \overrightarrow{R}$, and $(\overrightarrow{P} + \overrightarrow{Q}) + \overrightarrow{R}$. It is a parallelogram since each of a pair of opposite sides is parallel to $\overrightarrow{R}$ and has length equal to the length of $\overrightarrow{R}$. Similarly the terminal points of $\overrightarrow{R}$, $\overrightarrow{P} + \overrightarrow{Q}$, $\overrightarrow{Q} + \overrightarrow{R}$, and $\overrightarrow{P} + (\overrightarrow{Q} + \overrightarrow{R})$ are vertices of a parallelogram (opposite sides equal in length and parallel to $\overrightarrow{P}$). Thus the two parallelograms are identical and the fourth vertices must coincide.

105 A nicer proof depends on the one-to-one correspondence between points in the plane and ordered pairs of real numbers. It appears in the solution in Exercise 17, Section 3-6.

THEOREM 3-4. The vectors $(rs)\overline{P}$ and $r(s\overline{P})$ both have terminal point X such that $d(O,X) = rs\, d(O,P)$.

Exercises 3-4

1. (a) Show that: $\overrightarrow{B} + (\overrightarrow{A} - \overrightarrow{B}) = \overrightarrow{A}$

If $\overrightarrow{B} + (\overrightarrow{A} - \overrightarrow{B}) = \overrightarrow{A}$,

then $\overrightarrow{B} + (-\overrightarrow{B} + \overrightarrow{A}) = \overrightarrow{A}$

$\big(\overrightarrow{B} + (-\overrightarrow{B})\big) + \overrightarrow{A} = \overrightarrow{A}$

and $\overrightarrow{A} = \overrightarrow{A}$.

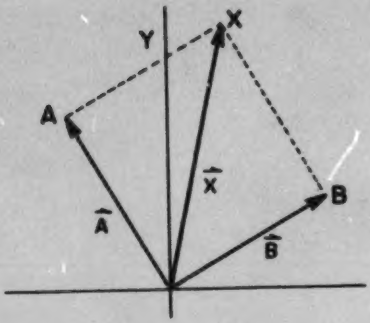
Since this last statement is true, the steps can be reversed to prove that $\overrightarrow{B} + (\overrightarrow{A} - \overrightarrow{B}) = \overrightarrow{A}$.

(b) If $(\overrightarrow{A} - \overrightarrow{B}) + \overrightarrow{B} = \overrightarrow{A}$,

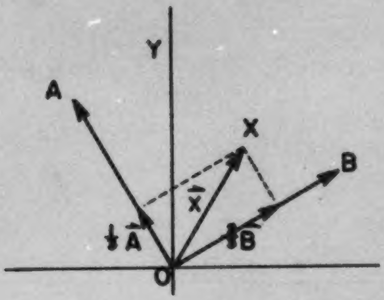
then $\overrightarrow{A} + \big((-\overrightarrow{B}) + \overrightarrow{B}\big) = \overrightarrow{A}$

and $\overrightarrow{A} = \overrightarrow{A}$. (See remark in part (a))

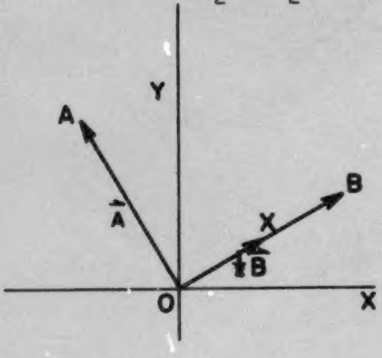
2. (a) $\overset{\frown}{X} = 1 \cdot \overset{\frown}{A} + 1 \cdot \overset{\frown}{B}$


X
Y
A
$\overset{\frown}{X}$
B
$\overset{\frown}{A}$
$\overset{\frown}{B}$

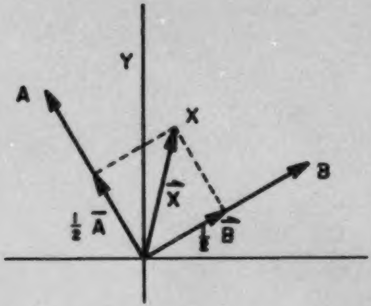
(b) $\overset{\frown}{X} = \frac{1}{3}\overset{\frown}{A} + \frac{2}{3}\overset{\frown}{B}$


Y
A
X
B
$\overset{\frown}{X}$
$\frac{1}{3}\overset{\frown}{A}$
$\frac{2}{3}\overset{\frown}{B}$
O

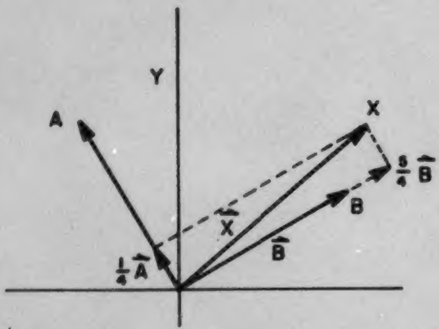
(c) $\overset{\frown}{X} = 0 \cdot \overset{\frown}{A} + \frac{1}{2}\overset{\frown}{B} = \frac{1}{2}\overset{\frown}{B}$


Y
A
$\overset{\frown}{A}$
B
X
$\frac{1}{2}\overset{\frown}{B}$
O
X

(d) $\overset{\frown}{X} = \frac{1}{2}\overset{\frown}{A} + \frac{1}{2}\overset{\frown}{B}$

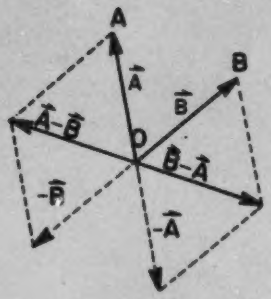

Y
A
X
B
$\overset{\frown}{X}$
$\frac{1}{2}\overline{A}$
$\frac{1}{2}\overset{\frown}{B}$

(e) $\overset{\frown}{X} = \frac{1}{4}\overset{\frown}{A} + \frac{5}{4}\overset{\frown}{B}$


Y
A
X
$\frac{5}{4}\overset{\frown}{B}$
$\overset{\frown}{X}$
B
$\overset{\frown}{B}$
$\frac{1}{4}\overset{\frown}{A}$

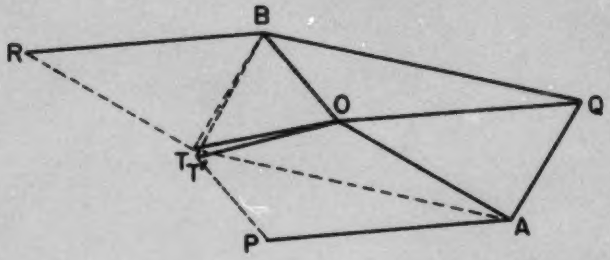
X is on $\overleftrightarrow{AB}$ when the sum of $p + q$ is 1 .

3. (a)



(b) $\vec{A} - \vec{B} = r(\vec{B} - \vec{A})$ for $r = -1$

4\.



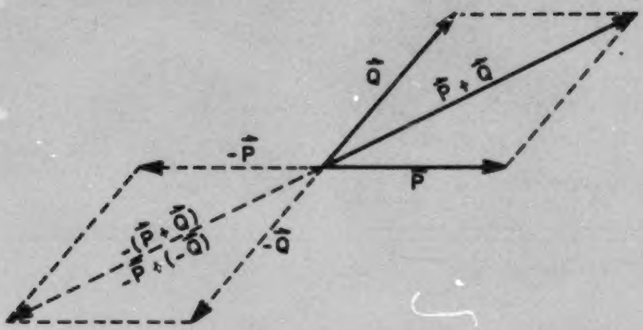
Let O be the origin and points P, Q, R determine vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$.
Let A be the vertex opposite O in the parallelogram determined by $\vec{R}$ and $\vec{Q}$, i.e., $\vec{A} = \vec{P} + \vec{Q}$.
Let B be the vertex opposite O in the parallelogram determined by $\vec{Q}$ and $\vec{R}$, i.e., $\vec{B} = \vec{Q} + \vec{R}$.

Let $\vec{T} = \vec{A} + \vec{R}$ and $\vec{T'} = \vec{P} + \vec{B}$

$= (\vec{P} + \vec{Q}) + \vec{R}$ $\qquad = \vec{P} + (\vec{Q} + \vec{R})$

We wish to prove $\vec{T} = \vec{T'}$. It is enough to show that T and T' coincide.
By using Exercises 3-3, Problem 17, $\overline{AT} \parallel \overline{OR} \parallel \overline{QB}$ and $d(A,T) = d(O,B) = d(Q,B)$.
Thus ATBQ is a parallelogram so $\overline{BT} \parallel \overline{QA}$ and $d(B,T) = d(B,T)$.
By construction of A, $\overline{OP} \parallel \overline{QA}$ and $d(OP) = d(QA)$.
By construction of T', $\overline{BT'} \parallel \overline{OP}$ and $d(B,T') = d(O,P)$.
Therefore $\overline{BT} \parallel \overline{BT'}$ and $d(B,T) = d(B,T')$.
So we must have $\overline{BT} = \overline{BT'}$.
Whence $T = T'$ and $\vec{T} = \vec{T'}$. Q.E.D.

5. $-(\vec{P} + \vec{Q}) = -\vec{P} - \vec{Q}$
$\qquad\qquad = -\vec{P} + (-\vec{Q})$



6. If $(-r)\vec{P} = r(-\vec{P})$,
then $(-r)\vec{P} = r[(-1)(\vec{P})]$
$(-r)\vec{P} = (r)(-1)(\vec{P})$
and $(-r)\vec{P} = (-r)\vec{P}$.

Since this last statement is true, the steps can be reversed to prove that $(-r)\vec{P} = r(-\vec{P})$.

3-5. Characterization of the Point on a Line.

109 In the proof of the distributive laws (Theorem 3-6), we left two items as unfinished business. The first was the proof in the case where $\vec{P}$ and $\vec{Q}$ are collinear and have opposite senses of direction.

In this case, assume $|\vec{P}| > |\vec{Q}|$. Then:

(1) By the same definition we used earlier, $\vec{P} + \vec{Q}$ has the same direction as $\vec{P}$ and has magnitude $|\vec{P}| - |\vec{Q}|$.

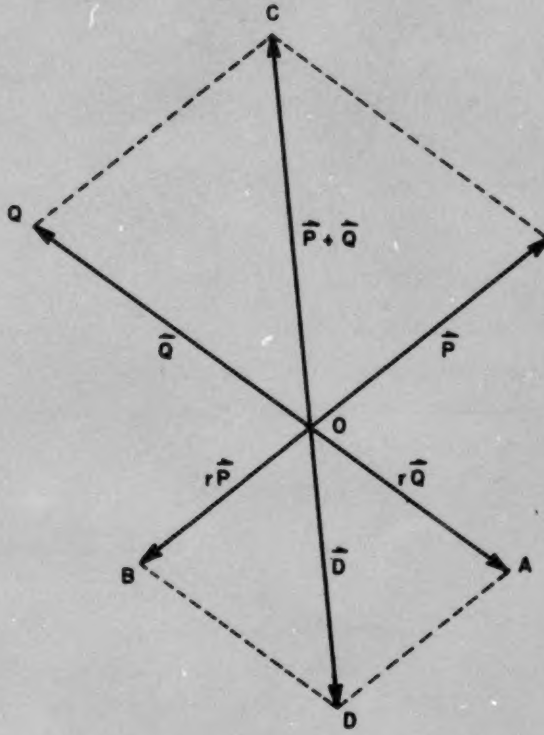
(2) If $r > 0$, then $r(\vec{P} + \vec{Q})$ has the same direction as $(\vec{P} + \vec{Q})$, and, by (1) above, the same direction as $\vec{P}$. The magnitude of $r(\vec{P} + \vec{Q}) = |r(\vec{P} + \vec{Q})| = r|\vec{P} + \vec{Q}|$ and is, by (1) above, equal to $r(|\vec{P}| - |\vec{Q}|)$. The distributive law gives the magnitude as $r|\vec{P}| - r|\vec{Q}|$.

(3) We now consider $r\vec{P}$ and $r\vec{Q}$, which, since $r > 0$, have the same directions respectively as $\vec{P}$ and $\vec{Q}$. By our hypothesis, $\vec{P}$ and $\vec{Q}$ have opposite senses of directions, and therefore so do $r\vec{P}$ and $r\vec{Q}$. Since we have assumed $|\vec{P}| > |\vec{Q}|$, we have $r|\vec{P}| > r|\vec{Q}|$, and, therefore $|r\vec{P}| > |r\vec{Q}|$.

(4) Our definition for the sum of vectors now requires that $r\vec{P} + r\vec{Q}$ have the same direction as $r\vec{P}$ and this is the same direction as $\vec{P}$. The same definition requires that the magnitude of $r\vec{P} + r\vec{Q}$. be $|r\vec{P}| - |r\vec{Q}|$; but this latter expression can be written as $r(|\vec{P}| - |\vec{Q}|)$.

(5) Since we have shown that the vectors $r(\vec{P} + \vec{Q})$ and $r\vec{P} + r\vec{Q}$ have the same magnitude and the same sense of direction, we have shown that they are equal.

The second item we did not discuss concerned the proof when $r < 0$. In this case, our figure must be changed to the following:



$\vec{A} = r\vec{Q}$

$\vec{B} = r\vec{P}$

Since $r < 0$, $r\vec{P}$ and $r\vec{Q}$ have directions opposite those of $\vec{P}$ and $\vec{Q}$ respectively. The proof for the case $r > 0$ in the text will need to be modified as follows in order to hold when $r < 0$.

In step (1) , since r is negative and the absolute values positive, $|\vec{A}| = -r|\vec{Q}|$ and $|\vec{B}| = -r|\vec{P}|$.

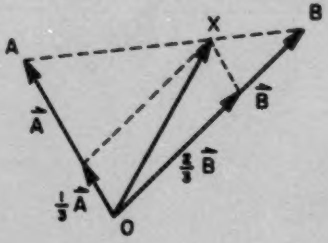
In step (2) $\dfrac{|\vec{B}|}{|\vec{A}|} = \dfrac{-r|\vec{P}|}{-r|\vec{Q}|} = \dfrac{|\vec{P}|}{|\vec{Q}|}$.

In step (5), $d(O,D) = |rd(O,C)|$,

$$|\overrightarrow{D}| = |r\overrightarrow{C}| .$$

In step (6), since the vectors are in opposite directions, $\overrightarrow{D} = r\overrightarrow{C}$.

110 When teaching this section, we would recommend that at first specific numbers be used for p and q . As an example, consider the line $\overleftrightarrow{AB} = \{X : \overrightarrow{X} = p\overrightarrow{A} + q\overrightarrow{B}$, where $p + q = 1\}$. Let $p = \frac{1}{3}$, $q = \frac{2}{3}$. Then $\overrightarrow{X} = \frac{1}{3}\overrightarrow{A} + \frac{2}{3}\overrightarrow{B}$.


B
X
A
$\overrightarrow{B}$
$\overrightarrow{A}$
$\frac{2}{3}\overrightarrow{B}$
$\frac{1}{3}\overrightarrow{A}$
O

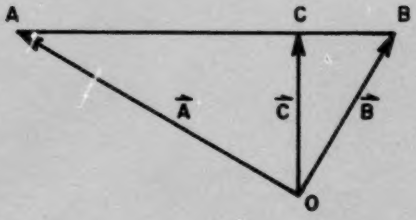
Take any vectors $\overrightarrow{A}$ and $\overrightarrow{B}$. Find the sum of $\frac{1}{3}\overrightarrow{A}$ and $\frac{2}{3}\overrightarrow{B}$ and verify, by construction, that X lies on $\overline{AB}$. Then let $p = \frac{4}{3}$ and $q = -\frac{1}{3}$ and see if the statement still holds.

Such experiences will help the students visualize what is really taking place.

111 In Chapter 2, a formula was developed for finding the coordinates of a point which divides a line segment in a given ratio. A comparable result for vectors is derived in Theorem 3-8. It may be of interest to the student to compare the derivations and the applications of the results.

Exercises 3-5

1.


A
C
B
$\overrightarrow{A}$
$\overrightarrow{C}$
$\overrightarrow{B}$
O

(a) if $\overrightarrow{A}$ is the zero vector, $\overrightarrow{C} = q\overrightarrow{B}$ and
if $\overrightarrow{B}$ is the zero vector $\overrightarrow{C} = p\overrightarrow{A}$

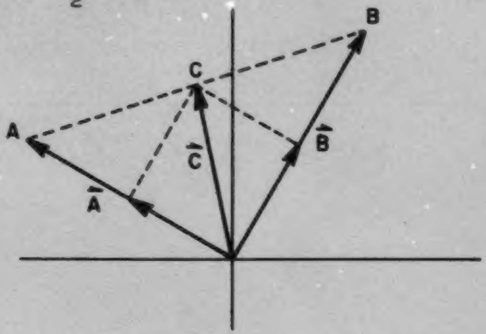
(b) if $\overrightarrow{C} = \overrightarrow{A}$, $p = 1$, $q = 0$

(c) (i) if $p > 0$, and $q > 0$, the terminal point of $\vec{C}$ lies in $\overline{AB}$.

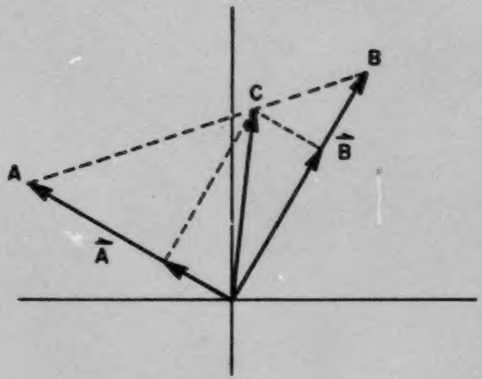
(ii) if $p < 0$, the terminal point of $\vec{C}$ lies on $\overleftrightarrow{AB}$ but rot on $\overline{AB}$.

(iii) if $p = 0$, $\vec{C} = q\vec{B}$ and $\vec{C}$ lies on $\overleftrightarrow{OB}$.

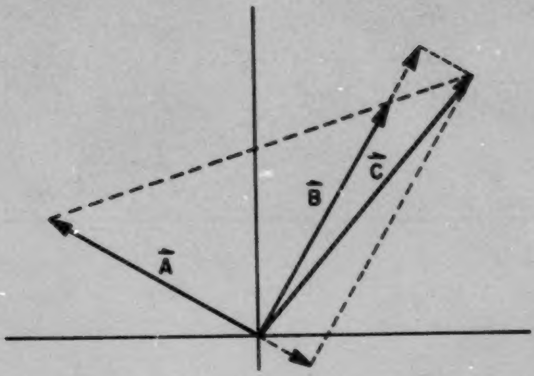
(d) (i) $p = q = \frac{1}{2}$


B
C
A
$\vec{C}$
$\vec{B}$
$\vec{A}$

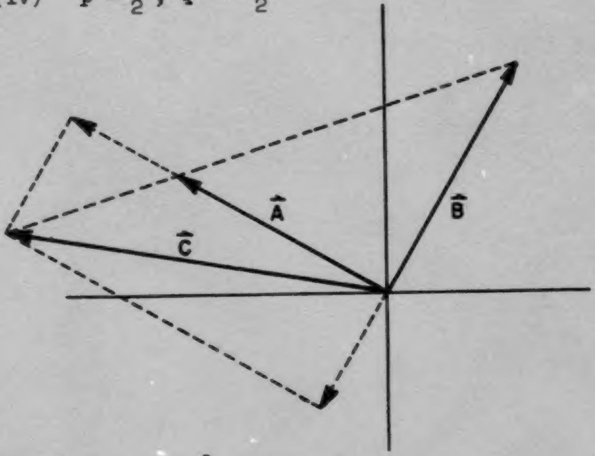
(ii) $p = \frac{1}{3}$, $q = \frac{2}{3}$


B
C
A
$\vec{B}$
$\vec{A}$

(iii) $p = -\frac{1}{4}$, $q = \frac{5}{4}$


$\vec{B}$
$\vec{C}$
$\vec{A}$

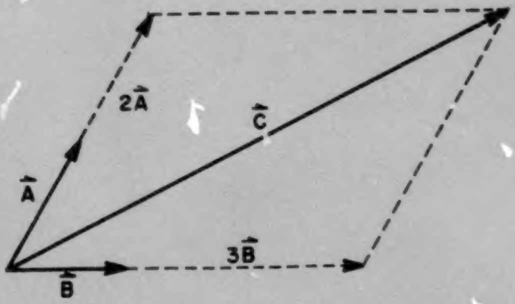
(iv) $p = \frac{3}{2}$, $q = -\frac{1}{2}$


$\vec{A}$
$\vec{B}$
$\vec{C}$

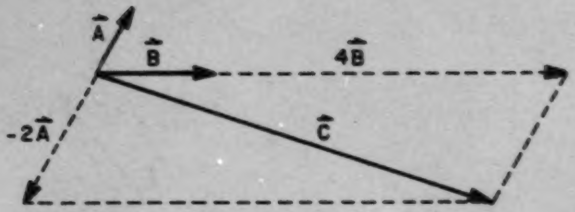
2. (a) $n = \frac{2}{5}$ and $m = \frac{3}{5}$

   (b) $m = \frac{5}{2}$ and $n = -\frac{3}{2}$

3. (a)


$2\vec{A}$
$\vec{C}$
$\vec{A}$
$3\vec{B}$
$\vec{B}$

(b)



4. Prove: $(r + s)\vec{P} = r\vec{P} + s\vec{P}$

We note that $(r + s)\vec{P} \parallel r\vec{P} + s\vec{P}$

Case 1: $r > 0$, $s > 0$.

$r > 0$, $s > 0$ imply $r + s > 0$. Thus $(r + s)\vec{P}$ and $r\vec{P} + s\vec{P}$ have the same sense of direction, and

$$|(r + s)\vec{P}| = (r + s)|\vec{P}| = r|\vec{P}| + s|\vec{P}| = |r\vec{P}| + |s\vec{P}| = |r\vec{P} + s\vec{P}| .$$

Case 2: $r > 0$, $s < 0$. $r > |s|$

Then $r + s > 0$ and $|(r + s)\vec{P}| = (r + s)|\vec{P}| = (r - |s|)|\vec{P}| =$ $r|\vec{P}| - |s|\vec{P} = |r\vec{P}| - |s\vec{P}| = |r\vec{P} + s\vec{P}|$.

Case 3. $r > 0$, $s < 0$, $r < |s|$

Then $r + s < 0$ and $|(r + s)\vec{P}| = -(r + s)|\vec{P}| = (-|r| + |s|)|\vec{P}| =$ $-|r|\vec{P} + |s|\vec{P} = -|r\vec{P}| + |s\vec{P}| = |r\vec{P} + s\vec{P}|$.

Case 4: $r > 0$, $s < 0$, $r = |s|$

$|(r + s)\vec{P}| = 0$ and $|r\vec{P} + s\vec{P}| = 0$

Case 5: $r = 0$ or $s = 0$. The proof follows from the definition of scalar multiplication.

3-6. Components.

113 The notation introduced in this section simplifies vector manipulations. A component is itself a real number and not a vector.

What is actually done in this section is to establish an isomorphism between vectors with certain operations and ordered pairs of real numbers for which certain operations are defined. This leads eventually to vector spaces which are characterized abstractly by postulating the basic properties exhibited in this treatment. A set of postulates for a vector space can be found in SMSG Intermediate Mathematics, page 678-682 or any text on modern algebra or linear algebra.

Since the origin-vector is unique, the vector [a,b] equals the vector [c,d] if and only if $a = c$ and $b = d$. This description of equality is used throughout the rest of the text and in many problems.

115 Part of the material presented earlier on the topic of linear combinations (See pages 108-109) is especially pertinent here. The unit vectors $i = [1,0]$ and $j = [0,1]$ in two dimensions and $i = [1,0,0]$, $j = [0,1,0]$ and $k = [0,0,1]$ in three dimensions are used in most applications of vector analysis. The i, j, k vectors are discussed in Chapter 8.

Exercises 3-6

1. (a) [7,3]   (e) [-5,-6]
   (b) [-1,-1]   (f) [-5,-6]
   (c) [20,24]   (g) [10,9]
   (d) [-20,-24]   (h) [14,-3]

2. (a) (1) [1,5]   (4) [2,-16]
   (2) [11,-8]   (5) [12,-22]
   (3) [13,-7]   (6) [-10,76]

   (b) (1) $\overrightarrow{X} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C} = [0,-2]$

   (2) $\overrightarrow{X} = \frac{1}{5}(2\overrightarrow{A} + 3\overrightarrow{B} - 4\overrightarrow{C}) = [-1,-\frac{4}{5}]$

   (3) $\overrightarrow{X} = \overrightarrow{C} - \frac{2}{3}\overrightarrow{A} + \frac{2}{3}\overrightarrow{B} = [-\frac{2}{3}, \frac{31}{3}]$

   (4) $\overrightarrow{X} = \frac{1}{3}(\overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A}) = [-\frac{2}{3}, \frac{14}{3}]$

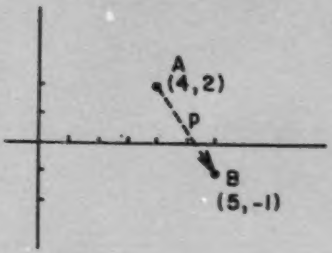
   (5) $\overrightarrow{X} = -2\overrightarrow{C} - 3\overrightarrow{B} = [-1,-24]$

   (6) $\overrightarrow{X} = -\frac{1}{3}\overrightarrow{A} - \frac{1}{2}\overrightarrow{B} = [-\frac{1}{2}, -\frac{4}{3}]$

3. 1
   0

4. (a) $\sqrt{2}$
   (b) 5
   (c) $\sqrt{a^2 + b^2}$
   (d) 1

5.



$\vec{A} = 4i + 2j$

$\vec{B} = 5i - j$

$\vec{p} = \vec{B} - \vec{A} = (5-4)i + (-1-2)j$

$= i - 3j$

6. $\vec{O} = 0 \cdot \vec{X} + 0 \cdot \vec{Y}$

7. The midpoint of the line segment joining (2,5) and (5,8) is $(\frac{7}{2}, \frac{13}{2})$

$$\vec{p} = \frac{7}{2}i + \frac{13}{2}j$$

8. (a) $\vec{p} = \frac{\sqrt{3}}{2}i + \frac{1}{2}j$

   (b) $\vec{q} = \frac{\sqrt{3}}{2}i - \frac{1}{2}j$

   (c) $\vec{r} = \frac{4}{5}i - \frac{3}{5}j$

9. (a) $x = \frac{-13}{6}$ $\quad y = \frac{23}{6}$

   (b) $x = \frac{-1}{5}$ $\quad y = \frac{4}{5}$

   (c) $x = \frac{27}{13}$ $\quad y = \frac{8}{13}$

   (d) $x = r$ $\quad y = \frac{-1-r}{2}$ for each real number. The real numbers form an infinite set.

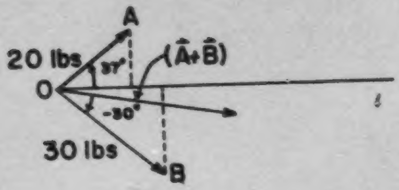
10. (a) $[a,b] = a[1,0] + b[0,1]$

   (b) $[a,b] = \frac{a+b}{2}[1,1] + \frac{b-a}{2}[-1,1]$

   (c) $[a,b] = -b\sqrt{2}\,[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}] + (b-a)[-1,0]$

11. $T_x = 25\sqrt{3}$ lbs. ≈ 43.3 lbs. $T_y = 25$ lbs.

12. Letting 1 lb. correspond to 1 unit, set up a coordinate system.



$\hat{A} = [A_x, A_y] = [|\hat{A}| \cos 37^\circ, |\hat{A}| \sin 37^\circ]$

$= [20 \cdot \frac{4}{5}, 20 \cdot \frac{3}{5}]$

$\hat{B} = [B_x, B_y] = [|\hat{B}| \cos(-30^\circ), |\hat{B}| \sin(-30^\circ)]$

$= [30\frac{\sqrt{3}}{2}, 30(-\frac{1}{2})]$

$\hat{A} + \hat{B} = [16, 12] + [15\sqrt{3}, -15] = [16 + 15\sqrt{3}, -3] \approx [42, -3]$.

13. (a) 24° , below x-axis in 4th quadrant. The components of the second vector, $\hat{B} = [26, -12]$

(b) 32° from y-axis in 2nd quadrant. The components of the second vector, $\hat{B} = [-16, 30]$

14. $24^\circ 30'$

15. (a) 21.3 lbs. acting 3° north of west.

(b) 31.3 lbs. acting 2° north of west.

In part (a) the components are $[-15\sqrt{2}, 15\sqrt{2} - 20]$

In part (b) the components are $[-10 - 15\sqrt{2}, 15\sqrt{2} - 20]$

16. 14.6 lbs.

17. THEOREM 3-1. Let $\hat{P} = [a,b]$ $\hat{Q} = [c,d]$

$\hat{P} + \hat{Q} = [a + c , b + d]$ and $\hat{Q} + \hat{P} = [c + a , b + d]$

But addition in the real numbers is commutative so $a + c = c + a$, $b + d = d + b$. Therefore $[a + c , b + d] = [c + a, d + b]$ which means $\hat{P} + \hat{Q} = \hat{Q} + \hat{P}$.

THEOREM 3-2. $\hat{P} = [a,b]$ $\hat{Q} = [c,d]$ $\hat{R} = [e,f]$

$(\hat{P} + \hat{Q}) + \hat{R} = [(a + c) + e , (b + d) + f]$

$\hat{P} + (\hat{Q} + \hat{R}) = [a + (c + e) , b + (d + f)]$

But addition in the reals is associative which means

$[(a + c) + e , (b + d) + f] = [a + (c + e) , b + (d + f)]$.

Hence, $(\hat{P} + \hat{Q}) + \hat{R} = \hat{P} + (\hat{Q} + \hat{R})$.

THEOREM 3-6. r and s are real numbers. $\overline{P} = [a,b]$, $\overline{R} = [c,d]$

(1) $r(\hat{P} + \hat{Q}) = r([a + c , b + d])$

$= [ra + rc , rb + rd]$

$= [ra, rb] + [rc, rd]$

$= r\hat{P} + r\hat{Q}$

(2) $(r + s)\overrightarrow{P} = (r + s)[a,b]$

$= [(r + s)a\,,\, (r + s)b]$

$= [ra + sa\,,\, rb + sb]$

$= [ra,rb] + [sa,sb]$

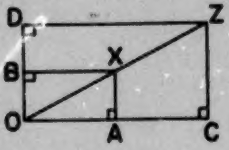
$= r\overrightarrow{P} + s\overrightarrow{P}$

18. THEOREM 3-10. If $\overrightarrow{X} = [a,b]$ and r is a real number, then $r\overrightarrow{X} = [ra,rb]$.

Case 1: $a = 0$. Then $\overrightarrow{X}$ lies along the y-axis. By definition, $r\overrightarrow{X}$ lies along the y-axis also with terminal point at rb . So $r\overrightarrow{X} = [r \cdot 0, rb] = [ra,rb]$.

Case 2: $b = 0$. By same argument $r\overrightarrow{X} = [ra,rb]$.

Case 3: $a \neq 0$ and $b \neq 0$.
We get $\triangle OXA \sim \triangle OZC$
and $\triangle OXB \sim \triangle OZD$.



Let $\overrightarrow{Z} = r\overrightarrow{X}$.
$X = (a,b)$

So $\dfrac{d(O,X)}{d(O,Z)} = \dfrac{d(O,A)}{d(O,C)} = \dfrac{d(O,B)}{d(O,D)} = \dfrac{1}{r}$

But $d(O,A) = a \quad d(O,B) = b$

Therefore $d(O,C) = ra \quad d(O,D) = rb$ and $Z = (ra,rb)$.

(alternatively)

If $\overrightarrow{X} = [a,b]$, define $\overrightarrow{A} = [a,0]$ $\overrightarrow{B} = [0,b]$ so that $\overrightarrow{X} = A + B$.

$r\overrightarrow{X} = r\overrightarrow{A} + r\overrightarrow{B}$

By Cases 1 and 2, $r\overrightarrow{A} = [ra,0]$, $r\overrightarrow{B} = [0,rb]$.

So $r\overrightarrow{X} = [ra,0] + [0,rb] = [ra,rb]$.

19. The vector representation of each set below is written so that if $r = 0$ we obtain $\overrightarrow{A}$ and if $r = 1$ we obtain $\overrightarrow{B}$.

(a) $\{[2 - 6r\,, 3 + 2r] : r$ is a real number$\}$
(b) $\{[1 + 2r\,, 3 + 6r] : r$ is a real number$\}$
(c) $\{[4\,, -7 + 9r] : r$ is a real number$]$
(d) $\{[2 + r] : r$ is a real number$]$
(e) $\{[-3 + 4r\,, 2 - 4r] : 0 \leq r \leq 1\}$
(f) $\{[1 + r] : 0 \leq r \leq 1\}$
(g) $\{[3 - 5r\,, 4 - r] : 0 \leq r \leq 1\}$
(h) $\{[1 - 4r\,, -2 + 4r] : 0 \leq r\}$
(i) $\{[2 - r] : 0 \leq r\}$
(j) $\{[3 - 5r\,, 4 - r] : 0 \leq r\}$
(k) $\{[-2 + 5r\,, 3 + r] : 0 \leq r\}$

(l) $\{[2 - r] : 0 \le r\}$

(m) $\{[3 - 5r , 4 - r] : 0 \ge r\}$

(n) $\{[-3 + 4r , 2 - 4r] : 0 < r < 1\}$

20. (a) $\overset{\rightharpoonup}{M} = [3,6]$, $\overset{\rightharpoonup}{T}_1 = [2,4]$, $\overset{\rightharpoonup}{T}_2 = [4,8]$

(b) $\overset{\rightharpoonup}{M} = [\frac{7}{2}, -\frac{9}{2}]$, $\overset{\rightharpoonup}{T}_1 = [\frac{4}{3}, -\frac{7}{3}]$, $\overset{\rightharpoonup}{T}_2 = [\frac{17}{3}, -\frac{20}{3}]$

(c) $\overset{\rightharpoonup}{M} = [\frac{a_1 + a_2}{2} , \frac{b_1 + b_2}{2}]$ $\overset{\rightharpoonup}{T}_1 = [\frac{2a_1 + b_1}{3} , \frac{2a_2 + b_2}{3}]$

$\overset{\rightharpoonup}{T}_2 = [\frac{a_1 + 2b_1}{3} , \frac{a_2 + 2b_2}{3}]$.

21. (a) $[2,8]$

(b) $[7]$

(c) $[0,0]$

(d) $[\frac{2}{3}, \frac{5}{2}]$

(e) $[\frac{39\pi + 2\sqrt{2}}{26(\sqrt{2} + \pi)} , \frac{26\pi + 24\sqrt{2}}{39(\sqrt{2} + \pi)}]$

(f) $[7]$

3-7. Inner Product.

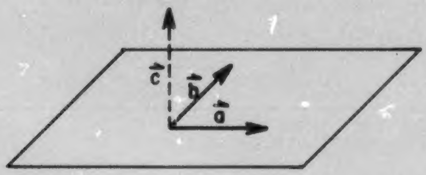
121 Although it is desirable algebraically to have some kind of vector multiplication, it is a little more difficult to introduce in a geometric framework. It would be possible to start by simply defining the inner product of two vectors by

$$[a_1,a_2] \cdot [b_1,b_2] = a_1b_1 + a_2b_2 .$$

This is quite satisfactory from the algebraic point of view, but does not connect very well with our development of vectors to this point. Hence a geometric approach is used by applying the law of cosines to the triangle formed by $\overset{\rightharpoonup}{X}$ and $\overset{\rightharpoonup}{Y}$. The definition of inner product is then made in terms of the resulting expression. The physical concept of work is one of the simplest applications of the inner product. It is included here to show that the inner product has relevance to a practical problem in science.

123 Theorem 3-13 establishes the connection between the geometric definition of inner product and its representation by components of the vectors. Either form can be used as indicated by a particular situation.

124 We did not present the vector product (or cross-product) $a \times b$ because some limitations had to be set for this chapter. The magnitude of $\overrightarrow{a} \times \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}| \sin \theta$; its direction lies along a line perpendicular to the plane determined by $\overrightarrow{a}$ and $\overrightarrow{b}$; and its sense of direction is determined by the motion of a right-hand screw when $\overrightarrow{a}$ is rotated into $\overrightarrow{b}$.



$\overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{b}$

You should note that $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$ because the sense of direction is reversed. Thus the commutative law fails. $|\overrightarrow{a} \times \overrightarrow{b}|$ is the area of the parallelogram with $\overrightarrow{a}$ and $\overrightarrow{b}$ as sides.

Your interested students may like to investigate this topic in a standard text on vector analysis.

Exercises 3-7

1. (a) 0
   (b) 0
   (c) 1
   (d) 1
   (e) 0
   (f) -7
   (g) ac + bd

2. (a) -11
   (b) -66
   (c) 48
   (d) -110
   (e) 29
   (f) -205
   (g) -76
   (h) 0
   (i) 347
   (j) 64

3. (a) 90°
   (b) 80°
   (c) 109°
   (d) 60°
   (e) 132°
   (f) 34°
   (g) 0°
   (h) 180°

4. (a) $|\overrightarrow{A}|^2 = 25$
   (b) $|\overrightarrow{B}|^2 = 169$

5. (a) $\frac{-16}{3}$

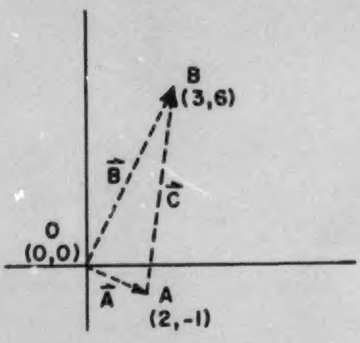
   (b) $\frac{16}{3}$

   (c) -3

   (d) 4

   (e) $-16i + 12j$; $16i - 12j$

6.



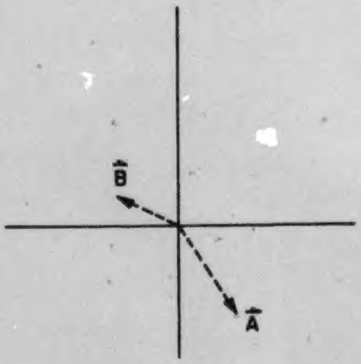
$\triangle AOB$ is a right $\triangle$.

If $\hat{C}$ is as shown,

$\hat{C} = \hat{B} - \hat{A}$

$\hat{C} = i + 7j$

7.



$\hat{A} = 2i - 3j$

$\hat{B} = 2i + j$

$\hat{A}\cdot\hat{B} = |\hat{A}|\cdot|\hat{B}| \cos\theta$

(a) $\hat{A}\cdot\hat{B} = (2)(-2) + (-3)(1) = -7$

From $\hat{A}\cdot\hat{B} = |\hat{A}||\hat{B}| \cos\theta$, we find that $\cos\theta = -.863$

$\therefore$ θ is approximately 150°

(b) Since $\hat{W} = \hat{F}\cdot\hat{S}$ and $\hat{F} = \hat{A} = 2i - 3j$

$\hat{S} = \hat{B} = 2i + 0j$,

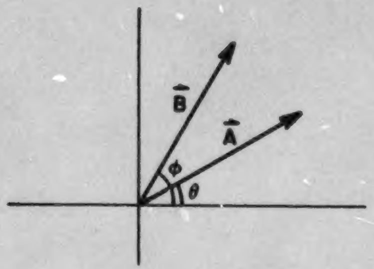
we have $\hat{W} = \hat{F}\cdot\hat{S} = a_1a_2 + b_1b_2$, and

$W = (2)(2) + (-3)(0) = 4$ (in proper units).

8. (a) 940 ft. lbs.
   (b) 8660 ft. lbs.

9. (a) 10.6 ft.
   (b) 588.2 ft.

10.



(a) $|\vec{A}| = \cos^2 \theta + \sin^2 \theta = 1$, $|\vec{B}| = \cos^2 \phi + \sin^2 \phi = 1$, and $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos \psi$ where ψ is the angle between $\overline{A}$ and $\overline{B}$.

(b) In this case $\psi = \phi - \theta$

$\therefore$ $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos (\phi - \theta) = 1 \cdot 1 \cdot \cos (\phi - \theta) = \cos (\phi - \theta)$.

Using components $\vec{A} \cdot \vec{B} = \cos \phi \cos \theta + \sin \phi \sin \theta$.

Thus $\cos (\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta$.

11. To show $-1 \leq \dfrac{\vec{X} \cdot \vec{Y}}{|\vec{X}| \cdot |\vec{Y}|} \leq 1$.

This expression is defined only if $\vec{X} \neq \vec{0}$ and $\vec{Y} \neq \vec{0}$. In this case $\vec{X} \cdot \vec{Y}$ is defined as $|\vec{X}|\,|\vec{Y}| \cos \theta$. Now $-1 \leq \cos \theta \leq 1$ for any angle, XOY , $|\vec{X}|\,|\vec{Y}| \neq 0$ so we may multiply through by

$$1 = \frac{|\vec{X}|\,|\vec{Y}|}{|\vec{X}|\,|\vec{Y}|} \text{ getting } -1 \leq \frac{\vec{X} \cdot \vec{Y}}{|\vec{X}|\,|\vec{Y}|} \leq 1.$$

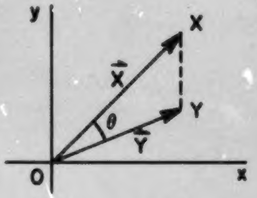
12. There is no associative law for inner products. The inner product of two vectors is a scalar.

3-8. Laws and Applications of the Inner (Dot) Product.

128 Most of the proofs of geometric theorems have been left for Chapter 4. These two proofs are given here to demonstrate that an abstract concept, such as the inner product of vectors, can be useful. The proof of the concurrence of the altitudes of a triangle is, we hope, impressive.

129 A bright student may ask why $\overleftrightarrow{BE}$ must intersect $\overleftrightarrow{CF}$ or why $\overleftrightarrow{AH}$ must intersect $\overleftrightarrow{BC}$. The answer is far from simple and involves a number of theorems involving the concepts of order, incidence, and betweenness. A careful treatment of such questions is given by E.E. Moise in his book Elementary Geometry from an Advanced Viewpoint. A careful non-vector proof of this theorem is in SMSG Geometry with Coordinates, p. 600-601.

131 A second derivation of the formula for the area of a triangle, $K = \frac{1}{2}|x_1y_2 - x_2y_1|$ is as follows:



(1) Consider $\triangle OXY$ and the realted non-zero vectors $\vec{X} = [x_1, x_2]$ and $\vec{Y} = [y_1, y_2]$ and the angle θ between them. Applying the trigonometric form for the area of a triangle, we have

$$K = \frac{1}{2}|\vec{X}||\vec{Y}| \sin\theta .$$

(2) Since $\vec{X}\cdot\vec{Y} = |\vec{X}||\vec{Y}| \cos\theta$, we have $|\vec{X}||\vec{Y}| = \frac{\vec{X}\cdot\vec{Y}}{\cos\theta}$, and

$$K = \frac{1}{2}(\vec{X}\cdot\vec{Y}) \tan\theta , \ \theta \neq \frac{\pi}{2}$$

(If the vectors are perpendicular, $K = \frac{1}{2}|\vec{X}||\vec{Y}|$)

(3) To write the result in terms of components, we observe the following:

(a) $\vec{X}\cdot\vec{Y} = x_1y_1 + x_2y_2$

(b) $\cos\theta = \frac{\vec{X}\cdot\vec{Y}}{|\vec{X}||\vec{Y}|} = \frac{x_1y_1 + x_2y_2}{\sqrt{x_1^2 + x_2^2}\ \sqrt{y_1^2 + y_2^2}}$

(c) $\sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{1 - \frac{(x_1y_1 + x_2y_2)^2}{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}}$

$$= \frac{\pm(x_1y_2 - x_2y_1)}{\sqrt{x_1^2 + x_2^2}\ \sqrt{y_1^2 + y_2^2}} = \frac{\pm(x_1y_2 - x_2y_1)}{|X||Y|}$$

(4) Thus $K = \frac{1}{2}|x_1y_2 - x_2y_1|$.

Exercises 3-8,9

1. $\vec{X} = [2,4]$ $\vec{Y} = [-1,-3]$, $t = 5$

$$(t\vec{X})\cdot\vec{Y} = t(\vec{X}\cdot\vec{Y}) = (\vec{X})\cdot(t\vec{Y})$$

$$[10,20]\cdot[-1,-3] = 5([2,4]\cdot[-1,-3]) = [2,4]\cdot[-5,-15]$$

$$-10 - 60 = 5(-2 - 12) = -10 - 60$$

$$-70 = -70 = -70$$

2. If $\vec{X} = [x_1,x_2]$ and $\vec{Y} = [y_1y_2]$, prove that $(t\vec{X})\cdot\vec{Y} = \vec{X}\cdot(t\vec{Y})$ for any scalar t .

Proof: $(t\vec{X})\cdot\vec{Y} = \vec{X}\cdot(t\vec{Y})$ if

$$[tx_1,tx_2]\cdot[y_1y_2] = [x_1,x_2][ty_1,ty_2] \text{ or}$$

$$tx_1y_1 + tx_2y_2 = tx_1y_1 + ty_1y_2 .$$

Since this last statement is true, the steps can be reversed to prove the original statement of the theorem.

3. To prove:

$$\vec{X}\cdot(a\vec{Y} + b\vec{Z}) = a(\vec{X}\cdot\vec{Y}) + b(\vec{X}\cdot\vec{Z}) \text{ , we note}$$

that $\vec{X}\cdot(a\vec{Y}) + \vec{X}\cdot(b\vec{Z}) = a(\vec{X}\cdot\vec{Y}) + b(\vec{X}\cdot\vec{Z})$ (Theorem 3-14a)

and $a(\vec{X}\cdot\vec{Y}) + b(\vec{X}\cdot\vec{Z}) = a(\vec{X}\cdot\vec{Y}) + b(\vec{X}\cdot\vec{Z})$ (Theorem 3-14b)

4. (a) $(\vec{A} + \vec{B})\cdot(\vec{A} - \vec{B}) = (\vec{A} + \vec{B})\cdot\vec{A} - (\vec{A} + \vec{B})\cdot\vec{B}$ (Theorem 3-14a)

$= (\vec{A}\cdot\vec{A}) + (\vec{B}\cdot\vec{A}) - (\vec{A}\cdot\vec{B}) - (\vec{B}\cdot\vec{B})$ (Theorem 3-14a)

$= |\vec{A}|^2 - |\vec{B}|^2$ (Commutative Property of Inner Product and the fact that $\vec{A}\cdot\vec{A} = |\vec{A}|^2$, $\vec{B}\cdot\vec{B} = |\vec{B}|^2$)

(b) Construction: Two lines are parallel or intersect at a point.

(1) Theorem 3-12 and Theorem 3-14a .

(2) Same reason.

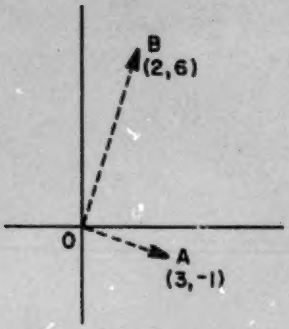
(3) Equality of real numbers and the commutative property.

(4) Additive property of equality.

(5) Theorem 3-14a and Theorem 12.

(6) $\vec{a}$ lies on $\overleftrightarrow{AD}$ and $(\vec{c} - \vec{b})$ lies on BC .

5.



$K = \frac{1}{2}|x_1y_2 - x_2y_1|$

$= \frac{1}{2}|18 + 2|$

$= \frac{1}{2}|20| = 10$

Check by alternate method : $\overline{OA} \perp \overline{OB}$ since $m_{\overline{OA}}$ is the negative reciprocal of $m_{\overline{OB}}$. $\therefore$ $\overline{OB}$ is an altitude of $\triangle OAB$.

$d(O,A) = \sqrt{10}$ and $d(O,B) = \sqrt{40}$

$A = \frac{1}{2}(\sqrt{10})(\sqrt{40}) = 10$

6. (a) $-\frac{7\sqrt{5}}{5}$

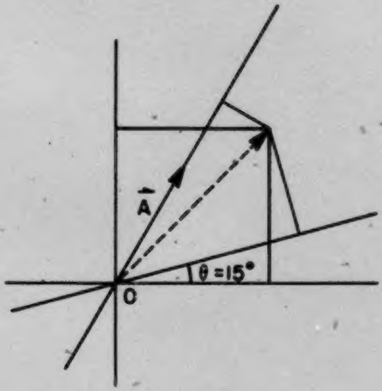
   (b) $-\frac{7\sqrt{13}}{13}$

7. (a) x direction, $15\sqrt{2}$

   y direction, $15\sqrt{2}$

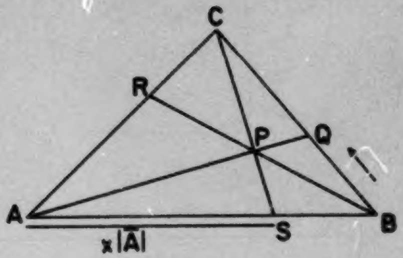
   (b) 26.0

   (c) 29.4



116

CHALLENGE PROBLEMS



1. Let P be any point not on △ ABC. Let $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, $\overleftrightarrow{CP}$ intersect sides $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AB}$ respectively at points Q, R, S.

To show $\dfrac{d(A,S)}{d(S,B)} \cdot \dfrac{d(B,Q)}{d(Q,C)} \cdot \dfrac{d(C,R)}{d(R,A)} = 1$

Take origin at A.

Then $\vec{R} = \dfrac{d(A,R)}{d(A,C)}\vec{C}$, $\vec{S} = \dfrac{d(A,S)}{d(A,B)}\vec{B}$

$\overleftrightarrow{CS}$ contains points $x\vec{C} + (1 - x)\vec{S} = x\vec{C} + (1 - x)\dfrac{d(A,S)}{d(A,B)}\vec{B}$ (1)

$\overleftrightarrow{BR}$ contains points $y\vec{B} + (1 - y)\vec{R} = y\vec{B} + (1 - y)\dfrac{d(A,R)}{d(A,C)}\vec{C}$ (2)

For intersection $y = (1 - x)\dfrac{d(A,S)}{d(A.B)}$ $x = (1 - y)\dfrac{d(A,R)}{d(A,C)}$

which reduces to $x = \dfrac{d(A,R) \cdot d(S,B)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}$

$$y = \frac{d(A,S) \cdot d(R,C)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}$$

Thus $\vec{P} = \dfrac{d(A,S) \cdot d(R,C)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\vec{B} +$

$$\frac{d(A,R) \cdot d(S,B)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\vec{C}$$

But Q is on $\overleftrightarrow{AP}$, so for some t we have

$$t\vec{P} = \vec{B} + \frac{d(Q,B)}{d(B,C)}(\vec{C} - \vec{B})$$

whence $t\left(\dfrac{d(A,S) \cdot d(R,C)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\right) = \dfrac{d(B,C) - d(Q,B)}{d(B,C)} = \dfrac{d(Q,C)}{d(B,C)}$ (3)

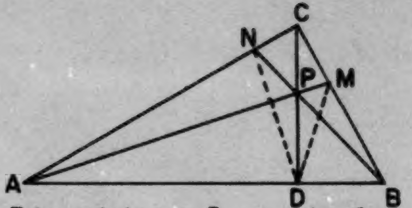
and $t\left(\dfrac{d(A,R) \cdot d(S,B)}{d(A,B) \cdot d(A,C) - d(A,S) \cdot d(A,R)}\right) = \dfrac{d(Q,B)}{d(B,C)}$ (4)

Substituting the expression for t obtained from (3) into (4) and simplifying we get

$$d(A,R) \cdot d(S,B) \cdot d(Q,C) \cdot d(B,C) \cdot d(B,C) \cdot d(A,S) \cdot d(R,C) \cdot d(Q,B)$$

which gives $\dfrac{d(A,S) \cdot d(C,R) \cdot d(Q,B)}{d(S,B) \cdot d(R,A) \cdot d(Q,C)} = 1$

2.



Consider $\triangle ABC$

$\overline{CD} \perp \overline{AB}$

P is a point on $\overline{CD}$

$\overleftrightarrow{PB}$ intersects $\overleftrightarrow{AC}$ at N

$\overleftrightarrow{PA}$ intersects $\overleftrightarrow{BC}$ at M

Take origin at D, x-axis along $\overleftrightarrow{AB}$, y-axis along $\overleftrightarrow{CD}$.

$A = [a,0]$ $B = [b,0]$ $C = [0,c]$ $P = [0,p]$

(This exercise considers only the case D strictly between A and B so that $a < 0 < b$ and $\frac{b}{a} \neq \frac{p}{c}$.)

If (x, y) is on $\overleftrightarrow{AC}$, then $y = \frac{c}{-a}(x - a)$

If (x, y) is on $\overleftrightarrow{PB}$, then $y = \frac{p}{-b}(x - b)$

Solving these to find coordinates of N we get

$$N = \left[\frac{ab(p - c)}{ap - bc}, \frac{cp(a - b)}{ap - bc}\right] = [N_x, N_y]$$

If (x,y) is on $\overleftrightarrow{BC}$, then $y = \frac{c}{-b}(x - b)$

If (x,y) is on $\overleftrightarrow{PA}$, then $y = \frac{p}{-a}(x - a)$

Solving for the coordinates of M we get

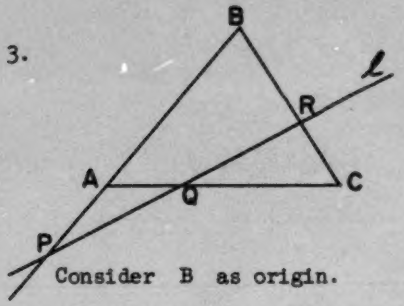
$$M = \left[\frac{ab(p - c)}{bc - ap}, \frac{ap(b - a)}{bc - ap}\right] = [M_x, M_y]$$

Because both $\angle NDC$ and $\angle MDC$ are smaller than 90° angles they are congruent if $|\sin \angle NDC|^2 = |\sin \angle MDC|$ for which it is enough that $|\sin \angle NDC|^2 = |\sin \angle MDC|^2$. But this follows from

$$|\sin \angle NDC|^2 = \frac{|N_x|^2}{d^2(ND)} = \frac{a^2b^2(c - p)^2}{(bc - ap)^2} \cdot \frac{(bc - ap)^2}{a^2b^2(c - p)^2 + c^2p^2(b - a)^2}$$

and $$|\sin \angle MDC|^2 = \frac{|M_x|^2}{d^2(MD)} = \frac{a^2b^2(p - c)^2}{(bp - ac)^2} \cdot \frac{(bc - ap)^2}{a^2b^2(c - p)^2 + c^2p^2(b - a)^2}.$$

3.



Let ℓ be any line which does not pass through any vertex of $\triangle ABC$. ℓ intersects $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ at P, Q, R, respectively. (This contains implicit assumption that ℓ is parallel to none of the sides of $\triangle ABC$.)

Consider B as origin.

$$\vec{P} = \frac{d(B,P)}{d(B,A)}\vec{A} \qquad \vec{R} = \frac{d(B,R)}{d(B,C)}\vec{C}$$

Q is on $\overleftrightarrow{AC}$ so for some x, $\vec{Q} = x\vec{A} + (1 - x)\vec{C}$

Q is on $\overleftrightarrow{PR}$ so for some y, $\vec{Q} = y\vec{P} + (1 - y)\vec{R}$

$$= y\frac{d(B,P)}{d(B,A)} A + (1 - y)\frac{d(B,R)}{d(B,C)} C$$

Hence $x = y\,\dfrac{d(B,P)}{d(B,A)}$ $\qquad (1 - x) = (1 - y)\,\dfrac{d(B,R)}{d(B,C)}$

From these we get

$$\vec{Q} = \frac{d(B,P)\cdot d(B,C)}{d(B,C)\cdot d(B,P) - d(B,A)\cdot d(B,R)}\vec{A} + \frac{d(B,R)\cdot d(A,P)}{d(B,C)\cdot d(B,P) - d(B,A)\cdot d(B,P)}\vec{C} \tag{1}$$

$\vec{Q}$ is a defined point only if the denominator is not zero, which is the condition that excludes ℓ parallel to a side.

Similarly we may write

$$\vec{Q} = \frac{d(Q,C)}{d(A,C)}\vec{A} + \frac{d(Q,A)}{d(A,C)}\vec{C} \tag{2}$$

Then the coefficients of $\vec{A}$ and $\vec{C}$ in (1) must be equal respectively to the corresponding coefficients in (2). From which we find

$$\frac{d(A,Q)}{d(Q,C)}\cdot\frac{d(C,R)}{d(R,B)}\cdot\frac{d(B,P)}{d(P,A)} = 1$$

4. (a) To show $(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)$

$$(x_1y_1 + x_2y_2)^2 = x_1^2y_1^2 + 2x_1y_1x_2y_2 + x_2^2y_2^2$$

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2$$

Thus we need to show that

$$2x_1y_1x_2y_2 \leq x_1^2y_2^2 + x_2^2y_1^2$$

But this is true because we always have

$$(x_1y_2 - x_2y_1)^2 = x_1^2y_2^2 - 2x_1y_2x_2y_1 + x_2^2y_1^2 \geq 0$$

(b) Let $\vec{X} = [x_1, y_1]$, $\vec{Y} = [x_2, y_2]$ in 2-space.

Then we write $(\vec{X} \cdot \vec{Y})^2 \leq |\vec{X}|^2 \cdot |\vec{Y}|^2$

(c) $(\vec{X} \cdot \vec{Y})^2 = |\vec{X}|^2 \cdot |\vec{Y}|^2$ if and only if $x_1y_2 = x_2y_1$, that is, if and only if $\vec{X} = r\vec{Y}$, $r \neq 0$

Review Exercises

1. (a) $\vec{X} = \vec{A} + \vec{B} - \vec{C} = [0,-2]$

(b) $\vec{X} = \frac{1}{5}(2\vec{A} + 3\vec{B} - 4\vec{C}) = [-1, -\frac{4}{5}]$

(c) $\vec{X} = \vec{C} - \frac{2}{3}\vec{A} + \frac{2}{3}\vec{B} = [-\frac{2}{3}, \frac{31}{3}]$

(d) $\vec{X} = \frac{1}{3}(\vec{B} + \vec{C} - \vec{A}) = [-\frac{2}{3}, \frac{14}{3}]$

(e) $\vec{X} = -2\vec{C} - 3\vec{B} = [-1,-24]$

(f) $\vec{X} = -\frac{1}{3}\vec{A} - \frac{1}{3}\vec{B} = [-\frac{1}{2}, -\frac{4}{3}]$

2. Prove: $\vec{A} + \vec{X} = \vec{0}$ is satisfied by
$\vec{X} = (-1)\vec{A} = -\vec{A}$

Proof:

$\vec{A} + \vec{X} = \vec{A} + (-1\vec{A})$. (Substitution)
$= \vec{A} + -\vec{A}$ (Definition of $(-1)\vec{A}$)
$= \vec{0}$. ($-\vec{A}$ is additive inverse of $\vec{A}$)

3. $(rs)\vec{P} = r(s\vec{P})$

Proof: $(rs)\vec{P}$ and $r(s\vec{P})$ are parallel and have the same sense of direction.

$$|(rs)\vec{P}| = |rs||\vec{P}| = |r||s||\vec{P}| = |r||s\vec{P}| = |r(s\vec{P})| .$$

4. (a) $[14,-3]$
(b) $[-7,16]$
(c) $[-2,17]$
(d) $[6,0]$
(e) $[14,10]$
(f) $[-18,-4]$

5. (a) $[-6,-2]$
(b) $[\frac{17}{5}, -\frac{12}{5}]$
(c) $[-\frac{1}{3}, -\frac{1}{3}]$
(d) $[0, -\frac{2}{3}]$
(e) $[-7,0]$
(f) $[-\frac{13}{6}, 0]$

6. (a) 0
(b) 0
(c) 21
(d) -36
(e) 0
(f) -38
(g) 243
(h) -4
(i) -192
(j) -11

7. (a) $2\sqrt{13}$
(b) $2\sqrt{13} + 3\sqrt{10}$
(c) $2\sqrt{13} + 3\sqrt{10}$
(d) $-\sqrt{13}$
(e) $\sqrt{26}$
(f) $\sqrt{226}$
(g) $5\sqrt{13}$
(h) 0
(i) 36
(j) 329
(k) 225
(l) 26
(m) 105
(n) 52

8. (a) $2(2i + 3j) + 3(3i - 2j) - (-i + 3j) = 4i + 6j + 9i - 6j + i - 3j$
$= 14i - 3j$
(b) $-7i + 16j$
(c) $-2i + 17j$
(d) $6i$
(e) $14i + 10j$
(f) $-18i - 4j$

9. (a) $\vec{X} = 6i - 2j$

(b) $2(2i + 3j) + 3(3i - 2j) = 4(-i + 3j) + 5(x_1 i + x_2 j)$

$$4i + 6j + 9i - 6j = -4i + 12j + 5x_1 i + 5x_2 j$$

$$17i - 12j = 5x_1 i + 5x_2 j$$

$$5x_1 = 17$$

$$x_1 = \frac{17}{5}$$

$$5x_2 = -12$$

$$x_2 = -\frac{12}{5}$$

$$\vec{X} = \frac{17}{5} i - \frac{12}{5} j$$

(c) $\vec{X} = -\frac{1}{3} i - \frac{1}{3} j$

(d) $2i + 3j + 2(x_1 i + x_2 j) = 3i - 2j - i + 3j - x_1 i - x_2 j$

$$2i + 3j + 2x_1 i + 2x_2 j = 3i - 2j - i + 3j - x_1 i - x_2 j$$

$$0i + 2j = -3x_1 i - 3x_2 j$$

$$x_1 = 0$$

$$-3x_2 = 2$$

$$x_2 = -\frac{2}{3}$$

$$\vec{X} = -\frac{2}{3} j$$

(e) $\vec{X} = -7i$

(f) $\vec{X} = -\frac{13}{6} i$

10. (a) $(2i + 3j) \cdot (3i - 2j) = (2)(3) + (3)(-2) = 0$

(b) $2(2i + 3j) \cdot 3(3i - 2j) = (4i + 6j) \cdot (9i - 6j) = (4)(9) + (6)(-6) = 0$

(c) 21

(d) -36

(e) 0

(f) -38

(g) $\big(3(2i + 3j) + 5(3i - 2j)\big) \cdot \big(3(3i - 2j) - 2(-i + 3j)\big)$

$$\big(6i + 9j + 15i - 10j\big) \cdot \big(9i - 6j + 2i - 6j\big)$$

$$(21i - j) \cdot (11i - 12j) = (21)(11) + (-1)(-12) = 243$$

(h) -4

(i) -192

(j) 36

11. (a) $m\angle ABC = 90$ (in degrees)

$m\angle BCD = 100$

$m\angle CDA = 55$

$m\angle DAB = 115$

(b) Area of $\triangle OAB = 9$

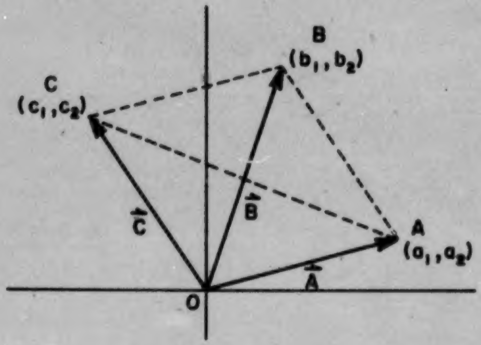
Area of $\triangle OBC = 8$

Area of $\triangle OAC = 7$

(c) Area of $\triangle ABC$ = Area of $\triangle OAB$ + Area of $\triangle OBC$ - Area $\triangle OAC$

$= 9 + 8 - 7 = 10$

12.



Area of $\triangle AOB = \frac{1}{2}|a_1b_2 - a_2b_1|$

Area of $\triangle BOC = \frac{1}{2}|b_1c_2 - b_2c_1|$

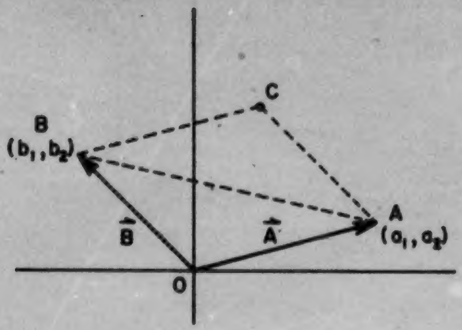
Area of $\triangle AOC = \frac{1}{2}|a_1c_2 - a_2c_1|$

From the diagram above: Area of $\triangle ABC$ = Area of $\triangle AOB$ + Area of $\triangle BOC$ - Area of $\triangle AOC$.

Area of $\triangle ABC = \frac{1}{2}|a_1b_2 - a_2b_1| + \frac{1}{2}|b_1c_2 - b_2c_1| - \frac{1}{2}|a_1c_2 - a_2c_1|$

Area of $\triangle ABC = \frac{1}{2}|a_1b_2 - a_2b_1 + b_1c_2 - b_2c_1 - a_1c_2 + a_2c_1|$

13.



Area of $\triangle AOB = \frac{1}{2}|a_1b_2 - a_2b_1|$

Area of $BOAC = 2(\text{Area of } \triangle AOB) = |a_1b_2 - a_2b_1|$

14. (a) $[-4, 7]$

(b) $[-4]$

(c) $[\frac{1}{2}, -\frac{9}{2}, \frac{11}{2}]$

(d) $[-15, \frac{23}{2}]$

15. (a) $\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$

$$= \{(2 - 3r, 3 - r) : 0 \le r \le 1\} \cup \{[-1 + 2r, 2 + 2r] : 0 \le r \le 1\}$$
$$\cup \{[1 + r, 3 + r] : 0 \le r \le 1\}$$

$$\text{Region } ABC = \{\vec{B} + r(\vec{A} - \vec{B}) + s(\vec{C} - \vec{B}) : 0 \le r \le 1, 0 \le s \le 1, r + s \le 1\}$$
$$= \{[-1 + 3r + 2s, 2 + r + 2s] : 0 \le r \le 1, 0 \le s \le 1, r + s \le 1\}$$

$$\text{Int.(Reg.ABC)} = \{\vec{B} + r(\vec{A} - \vec{B}) + s(\vec{C} - \vec{B}) : 0 < r < 1, 0 < s < 1, r + s < 1\}$$
$$= \{[-1 + 3r + 2s, 2 + r + 2s] : 0 < r < 1, 0 < s < 1, r + s < 1\}$$

(b) $[1,3] = [-1 + 3(\frac{1}{2}) + 2(\frac{1}{4}), 2 + (\frac{1}{2}) + 2(\frac{1}{4})]$ where we certainly have

$$0 < r = \frac{1}{2} < 1, \quad 0 < s = \frac{1}{4} < 1, \text{ and } r + s = \frac{3}{4} < 1$$

So $[3,1] \in$ Int.(Reg.ABC).

(c) $[1,1] = [-1 + 3r + 2s, 2 + r + 2s]$ if and only if $r = -\frac{3}{2}$, $s = -\frac{5}{4}$. So clearly $[1,1]$ does not satisfy the conditions to be in Region ABC.

(d) Segment $\overline{P_bP_c} = \{[1, 1 + 2t] : 0 \leq t \leq 1\}$

From graphical considerations, we show $\overline{P_bP_c}$ intersects $\overline{AB}$ which is a subset of $\triangle ABC$. The conditions $0 \leq r \leq 1$, $0 \leq t \leq 1$, $[2 - 3r, 3 - r] = [1, 1 + 2t]$ are met for $t = \frac{5}{6}$, $r = \frac{1}{3}$. Hence the segments intersect in the point $[1, \frac{8}{3}]$.

16. Region ABCD = Region BAD $\cup$ Region BDC $\cup$ Region BAC

$$= \{\bar{B} + r(\bar{A} - \bar{B}) + s(\bar{C} - \bar{B}) + t(\bar{D} - \bar{B}) : 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ r + s \leq 1,\ s + t \leq 1,\ r + t \leq 1\}$$

$$= \{[-1 + 3r + 2s + 3t,\ 2 + r + 2s + 2t] : 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ r + s \leq 1,\ s + t \leq 1,\ r + t \leq 1\}$$

Note: the commas indicate logical conjunction of the six individual conditions.

17. Region ABCD =

$$\{\bar{B} + r(\bar{A} - \bar{B}) + s(\bar{C} - \bar{B}) + t(\bar{D} - \bar{B}) : 0 \leq r \leq 1,\ 0 \leq s \leq 1,\ 0 \leq t \leq 1,\ r + s \leq 1,\ s + t \leq 1,\ r + t \leq 1\}$$

18. (a) 90°
(b) 97°
(c) 45°
(d) 61°

19. $\angle CAB = 90^\circ$
$\angle ABC = 45^\circ$
$\angle ACB = 45^\circ$

20. $\angle PSR = 135^\circ$
$\angle SRQ = 135^\circ$
$\angle RQP = 45^\circ$
$\angle QPS = 45^\circ$

Trapezoid

Chapter 4

PROOFS BY ANALYTIC METHODS

This is the first of what some students refer to as "fun" chapters. There is nothing new to learn in the sense that there are no new theorems or definitions. The students have accumulated a variety of tools; now they will see how these tools may be used. In spite of the groans and complaints one hears from the class, most students thoroughly enjoy this type of thing.

Our primary concern in this chapter is that each student develop a systematic approach to solving problems by coordinates or vectors. We feel that a satisfactory beginning can be made by writing analytic proofs of familiar geometric theorems. It is also our aim that, while he is operating with these analytic tools, each student realize and appreciate the power available in the application of these tools. These methods represent a tremendous advance in mathematics, and the students should be aware of their heritage.

After a discussion of three methods of proof--by rectangular coordinates, by vectors, by polar coordinates--the chapter culminates in a section where the student must make a conscious choice of method. In order that the student not be denied this valuable opportunity to develop mathematical maturity, the teacher must avoid the temptation to decide for the student. Every student is entitled to learn what happens when he makes a poor choice. Furthermore, his choice may be, for him, the best.

The exercise solutions are given in the form we think is the most natural; but, to follow the spirit of the text, the teacher should accept any presentation which is mathematically sound. Then if the teacher feels that the student could have produced a simpler or more direct proof by using another method, this could be pointed out.

4-2. Proofs Using Rectangular Coordinates.

This section, which is concerned with proofs using rectangular coordinates, may be skimmed or swiftly reviewed if the class has already covered this material in another course. Some time might be saved in this way since the time allotment for this chapter assumes that most of the students have had little or no experience in this area.

The techniques we recommend are developed by means of examples. Following Example 1, we have suggested a short outline of systematic steps a student may follow for the problems which seem particularly suited to rectangular coordinates. To facilitate the study of the examples, we suggest that each student copy the figure and supply coordinates for it as the proof proceeds.

Among other things, Example 1 illustrates a rather delicate choice the student must make. On one hand, he must select coordinates which make the figure perfectly general; on the other hand, he should choose coordinates which make use of the information given in the problem. If he does this improperly, in the first instance he may have a proof which is valid for only a special case; in the second instance he may have a very complicated proof where a simple one would suffice. Example 1 shows how the choice of coordinates may be improved without losing generality in the figure.

142 We use the fact that $d(A,C) = d(B,C)$ to show that $\overline{CD}$ has no slope.

$$\sqrt{b^2 + c^2} = \sqrt{(b - 2a)^2 + c^2} ,$$

or $$b^2 = b^2 - 4ab + 4a^2 .$$

Therefore, $$4ab = 4a^2 ,$$

and, if $a \neq 0$, then $a = b$ and $\overline{CD}$ is vertical.

142 Regarding the choice of coordinates for A and B in Figure 4-4, we deliberately chose "-a" to the right of "a" so that some students who need the reminder may note that -a does not necessarily represent a negative number. It means the opposite of a; hence, when a is negative, -a is positive.

To show that C lies on the y-axis, we note that

$$d(A,C) = d(B,C) ,$$

or $$\sqrt{(b - a)^2 + c^2} = \sqrt{(b - (-a))^2 + c^2} ,$$

or $$b^2 - 2ab + a^2 = b^2 + 2ab + a^2 .$$

Therefore, $$0 = 4ab,$$

and, if $a \neq 0$, then $b = 0$.

143 We justify the choice of abscissa for point C in Figure 4-5 in the following way. Let $D = (b,c)$ and $C = (d,c)$. Since $\overline{BC} \| \overline{AD}$, their slopes are equal. Thus

$$\frac{c}{d - a} = \frac{c}{b} , \ (a \neq d) ,$$

and $$b = d - a ,$$

or $$d = a + b .$$

We are dealing with well-known and previously proved properties of geometric figures; therefore, some confusion may exist in the class as to which of these properties may be assumed in choosing coordinates for the figure. Although the teacher is at liberty, of course, to set up his own "ground rules", we recommend that only those properties ascribed to geometric figures by their definitions or by the hypothesis be allowed when selecting the coordinates. For the purposes of this section, we have also allowed the theorems (after proof) of Exercises 4-2. The teacher is not bound by this. Our reason for the exception is to make it unnecessary for a student to prove the same thing in two separate exercises.

144 To complete the proof of Example 3, we note that for the conclusion, $d(A,C) = d(B,C)$, to be true, we must have

$$\sqrt{4a^2 + 4c^2} = \sqrt{4b^2 + 4c^2} .$$

This will hold is $a^2 = b^2$. From the hypothesis, we have $d(A,N) = d(B,M)$, or

$$\sqrt{(b - 2a)^2 + c^2} = \sqrt{(2b - a)^2 + c^2} .$$

This simplifies to

$$b^2 - 4ab + 4a^2 + c^2 = 4b^2 - 4ab + a^2 + c^2 ,$$

or $$3a^2 = 3b^2 ,$$

from which we have $a^2 = b^2$ as required.

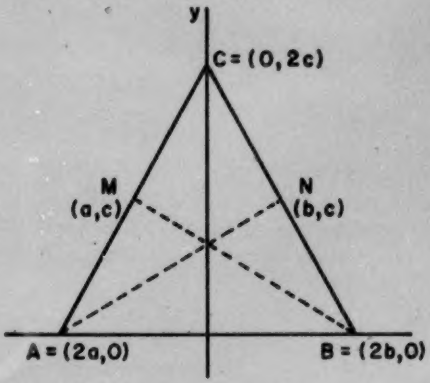
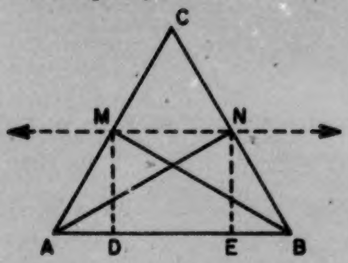


Figure 4-6

144 We include a sample synthetic proof for Example 3:



Hypothesis:

$\overline{BM}$ and $\overline{AN}$ are medians.

$\overline{BM} \cong \overline{AN}$.

Conclusion:

$\overline{AC} \cong \overline{BC}$.

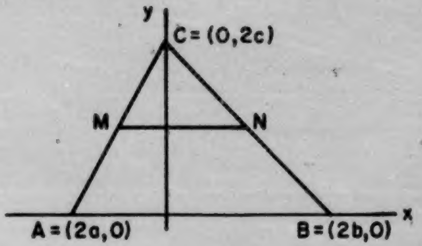
1. $\overline{BM}$ and $\overline{AN}$ are medians.	1. Hypothesis.
2. M is the midpoint of $\overline{AC}$; N is the midpoint of $\overline{BC}$.	2. Definition of median.
3. $\overleftrightarrow{MN} \parallel \overleftrightarrow{AB}$.	3. The line joining the midpoints of two sides of a triangle is parallel to the line containing the third side.
4. Introduce $\overline{MD}$ and $\overline{NE}$ perpendicular to $\overleftrightarrow{AB}$.	4. There is a unique perpendicular to a line from a point not on the line.
5. $\overline{MD} \cong \overline{NE}$.	5. Parallels are everywhere equidistant.
6. $\overline{BM} \cong \overline{AN}$.	6. Hypothesis.
7. $\triangle BMD$ and $\triangle ANE$ are right triangles.	7. Perpendiculars form right angles.
8. $\triangle BMD \cong \triangle ANE$.	8. Hypotenuse - leg theorem.
9. $\angle DBM \cong \angle EAN$.	9. Corresponding angles of congruent triangles are congruent.
10. $\overline{AB} \cong \overline{AB}$.	10. Reflective property of congruence for segments.
11. $\triangle ABM \cong \triangle BAN$.	11. S. A. S. theorem.
12. $\overline{AM} \cong \overline{BN}$.	12. Corresponding sides of congruent triangles are congruent.
13. $d(A,M) = d(B,N)$.	13. Definition of congruence.
14. $d(A,C) = d(B,C)$.	14. Definition of midpoint and multiplication property of equals.
15. $\overline{AC} \cong \overline{BC}$.	15. Definition of congruence.

145 It is not anticipated that the teacher will assign all of the parts of Exercises 4-2 to a single student. The excess exercises may be used for test items. It is suggested that exercises 10, 13, 16 be assigned to everyone. These theorems are proved by vector methods in the next section, and the students may profit from a comparison of the two methods of proof.

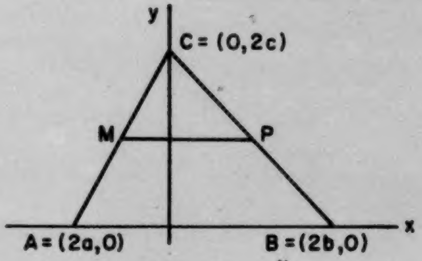
Exercises 4-2

(Note: Formal proofs are not presented here. We merely indicate the essentials of one possible solution for each problem.)

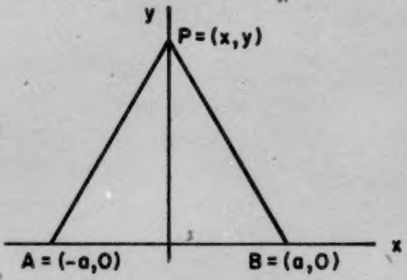
1. $M = (a,c)$; $N = (b,c)$.
   Slope of $\overline{MN} = 0$;
   slope of $\overline{AB} = 0$.
   $\therefore \overline{MN} \parallel \overline{AB}$.
   $d(M,N) = \sqrt{(a - b)^2} = |a - b|$.
   $d(A,B) = \sqrt{(2a - 2b)^2} = |2a - 2b|$
   $= 2|a - b|$.



2. $M = (a,c)$; since $\overline{MP} \parallel \overline{AB}$, $P = (x,c)$.
   P lies on $\overline{BC}$; therefore, slope of $\overline{PC}$ = slope of $\overline{BP}$; that is,
   $$\frac{-c}{x} = \frac{-c}{2b - x} .$$
   Thus, $x = b$ and $P = (b,c)$, the midpoint of $\overline{BC}$.



3. Part I. If $d(A,P) = d(B,P)$, then
   $\sqrt{(x + a)^2 + y^2} = \sqrt{(x - a)^2 + y^2}$,
   $x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2 + y^2$,
   and $4ax = 0$.
   Therefore, if $a \neq 0$, then $x = 0$ and P lies on the y-axis, the perpendicular bisector of $\overline{AB}$.



   Part II. If P lies on the perpendicular bisector of $\overline{AB}$, then $x = 0$ and $d(A,P) = \sqrt{a^2 + y^2} = \sqrt{(-a)^2 + y^2} = d(B,P)$.

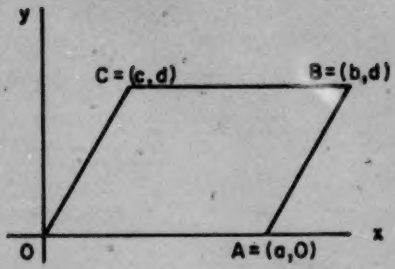
4. By definition $\overline{OC} \| \overline{AB}$ and their slopes are equal. Thus

$$\frac{d}{c} = \frac{d}{b - a}, \ (a \neq b),$$

and $b = a + c$. Therefore,

$$d(B,C) = \sqrt{a^2} = |a| = d(A,O)$$

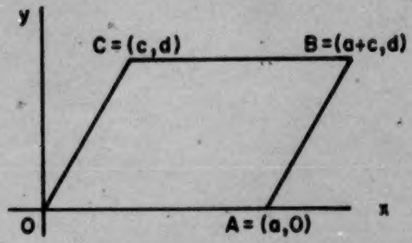
and $d(C,O) = \sqrt{c^2 + d^2} = d(B,A)$.



5. $B = (a + c, d)$ because $d(B,C) = d(O,A)$ and $\overline{BC} \| \overline{OA}$.

Slope of $\overline{OC} = \frac{d}{c}$ = slope of $\overline{AB}$;

therefore, $\overline{OC} \| \overline{AB}$.



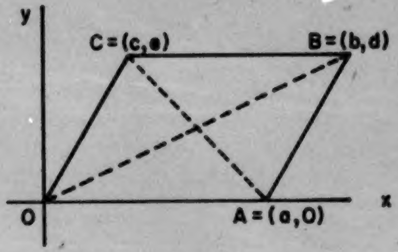
6. Midpoint of $\overline{OB} = \left(\frac{b}{2}, \frac{d}{2}\right)$;

midpoint of $\overline{AC} = \left(\frac{a + c}{2}, \frac{e}{2}\right)$.

Since $\left(\frac{b}{2}, \frac{d}{2}\right) = \left(\frac{a + c}{2}, \frac{e}{2}\right)$,

$b = a + c$ and $d = e$.

This satisfies the conditions for the theorem of Exercise 5.



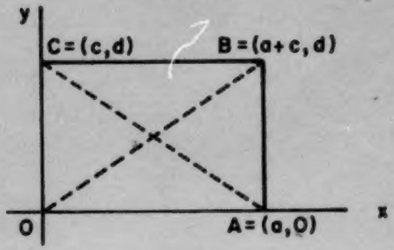
7. Since OABC is a parallelogram, it may have coordinates as in Exercise 5. Since $d(O,B) = d(A,C)$,

$$\sqrt{(a + c)^2 + d^2} = \sqrt{(a - c)^2 + d^2},$$

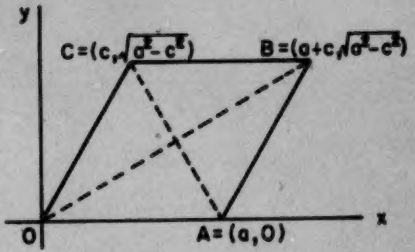
$$a^2 + 2ac + c^2 + d^2 = a^2 - 2ac + c^2 + d^2,$$

and $4ac = 0$.

If $a \neq 0$, then $c = 0$ and $B = (a,d)$; therefore, $\angle OAB$ is a right angle.



8. The coordinates shown in the figure take account of the fact that a rhombus is a parallelogram with congruent sides.

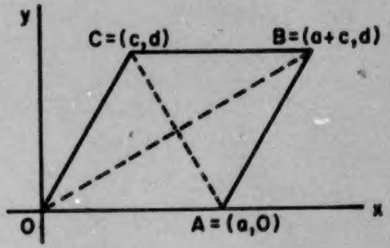

y
$C=(c,\sqrt{a^2-c^2})$
$B=(a+c,\sqrt{a^2-c^2})$
O
A = (a,0)
x

The slope of $\overline{AC}$ is $\frac{\sqrt{a^2 - c^2}}{c - a}$;

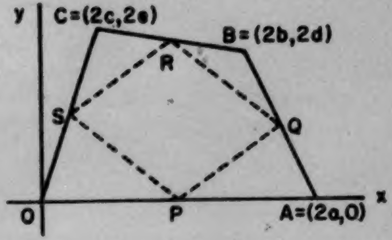
the slope of $\overline{OB}$ is $\frac{\sqrt{a^2 - c^2}}{a + c}$.

The product of the slopes is $\frac{a^2 - c^2}{c^2 - a^2} = -1$; hence, the diagonals are perpendicular.

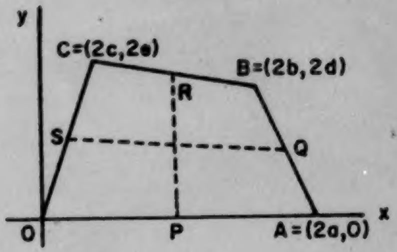
9. The slope of $\overline{AC} = \frac{d}{c - a}$; the slope of $\overline{OB} = \frac{d}{a + c}$. Since $\overline{AC} \perp \overline{OB}$, $\frac{d}{c - a} \cdot \frac{d}{a + c} = -1$. Therefore, $d^2 = a^2 - c^2$, or $a^2 = c^2 + d^2$. Hence, $|a| = \sqrt{c^2 + d^2} = d(O,C) = d(O,A)$.


y
C=(c,d)
B=(a+c,d)
O
A = (a,0)
x

10. $P = (a,0)$; $Q = (a + b,d)$;
$R = (b + c,d + e)$; $S = (c,e)$.
Slope of $\overline{PQ}$ = slope of $\overline{RS} = \frac{d}{b}$;
slope of $\overline{PS}$ = slope of $\overline{RQ} = \frac{e}{c - a}$.


y
C=(2c,2e)
B=(2b,2d)
R
S
Q
O
P
A=(2a,0)
x

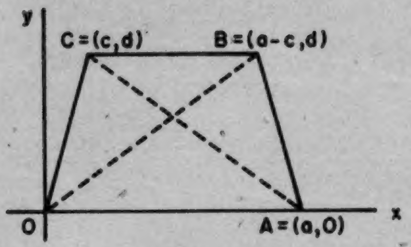
11. $P = (a,0)$; $Q = (a + b,d)$;
$R = (b + c,d + e)$; $S = (c,e)$.
Midpoint of $\overline{RP} = \left(\frac{a + b + c}{2}, \frac{d + e}{2}\right)$;
midpoint of $\overline{SQ} = \left(\frac{a + b + c}{2}, \frac{d + e}{2}\right)$.


y
C=(2c,2e)
B=(2b,2d)
R
S
Q
O
P
A=(2a,0)
x

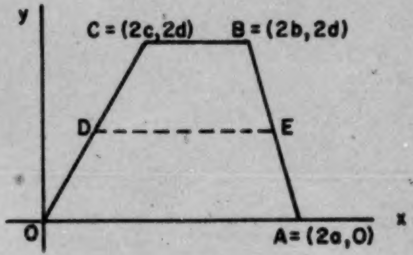
12. $d(A,C) = \sqrt{(c-a)^2 + d^2}$

$= \sqrt{(a-c)^2 + d^2}$

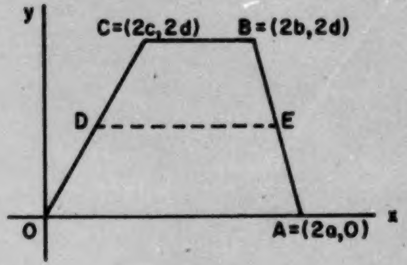
$= d(O,B)$.


y
C=(c,d)
B=(a-c,d)
O
A=(a,0)
x

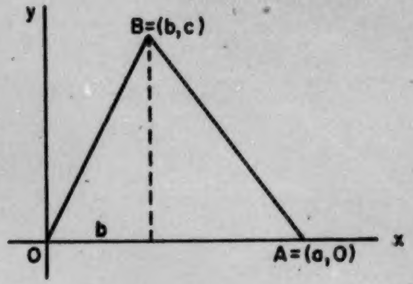
13. $D = (c,d)$; $E = (a+b,d)$.
Slope of $\overline{DE} = 0 =$ slope of $\overline{OA}$
and slope of $\overline{BC}$.
$d(O,A) = d(C,B) = 2a + 2b - 2c$
$= 2(a + b - c)$.
$d(D,E) = a + b - c$.


y
C=(2c,2d)
B=(2b,2d)
D
E
O
A=(2a,0)
x

14. $D = (c,d)$; let $E = (e,d)$.
Since E lies on $\overline{AB}$, the slope
of $\overline{BE} =$ the slope of $\overline{AE}$; hence,
$\frac{d}{2b - e} = \frac{d}{e - 2a}$, $2e = 2a + 2b$,
and $e = a + b$. Therefore
$E = (a + b,d)$, the midpoint of $\overline{AB}$.


y
C=(2c,2d)
B=(2b,2d)
D
E
O
A=(2a,0)
x

15. Let the acute angle be at O.
$(d(A,B))^2 = (b - a)^2 + c^2$
$= b^2 - 2ab + a^2 + c^2$.
Also $(d(O,B))^2 + (d(O,A))^2$
$- 2d(O,A)b$
$= (b^2 + c^2) + a^2 - 2ab$
$= b^2 - 2ab + a^2 + c^2$.

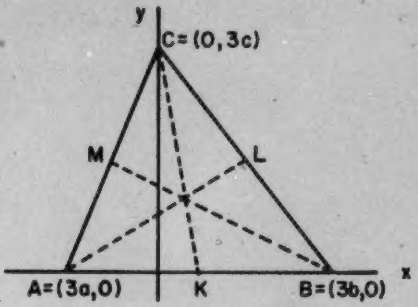

y
B=(b,c)
b
O
A=(a,0)
x

16. $K = \left(\frac{3a + 3b}{2}, 0\right)$;

$L = \left(\frac{3b}{2}, \frac{3c}{2}\right)$;

$M = \left(\frac{3a}{2}, \frac{3c}{2}\right)$.

The point $(a + b, c)$ divides each of $\overline{CK}$, $\overline{BM}$, and $\overline{AL}$ in the ratio 2:1.

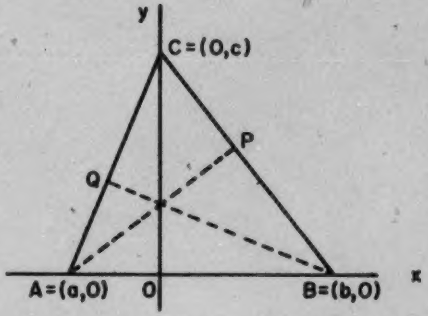


17. Since $\overline{AP} \perp \overline{BC}$, the slope of $\overline{AP} = \frac{b}{c}$; since $\overline{BQ} \perp \overline{AC}$, the slope of $\overline{BQ} = \frac{a}{c}$.

$\overleftrightarrow{AP} = \{(x,y):\ y = \frac{b}{c}(x - a)\}$;

$\overleftrightarrow{BQ} = \{(x,y):\ y = \frac{a}{c}(x - b)\}$.

Since the intersection must lie on the y-axis, $x = 0$, and the point is $(0, -\frac{ab}{c})$.



* 18. In the solution of this exercise we wish to make use of the proposition: The segment joining the center of a circle to the midpoint of a chord of the circle is perpendicular to the chord. We dispose of this proposition first.

Since $d(O,A) = d(O,B)$,

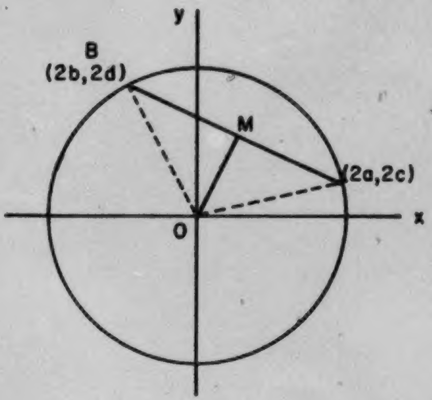
$\sqrt{4a^2 + 4c^2} = \sqrt{4b^2 + 4d^2}$,

or $a^2 + c^2 = b^2 + d^2$.

The slope of $\overline{AB} = \frac{c - d}{a - b}$;

the slope of $\overline{OM} = \frac{c + d}{a + b}$.

The product of these slopes is $\frac{c^2 - d^2}{a^2 - b^2}$, and, since

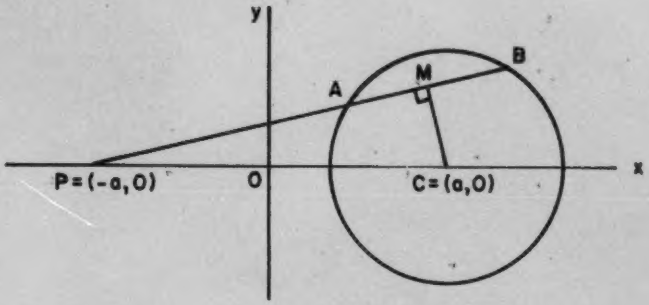


$a^2 + c^2 = b^2 + d^2$, $c^2 - d^2 = b^2 - a^2$.

Substituting in the product of the slopes obtained above, we have

$$\frac{b^2 - a^2}{a^2 - b^2} = -1 ;$$

therefore, $\overline{OM} \perp \overline{AB}$.



We return to the first problem and select a coordinate system as depicted in the figure. We have placed the origin at the midpoint of $\overline{PC}$, and we let $M = (x,y)$.

We then have $d(P,M) = \sqrt{(x + a)^2 + y^2}$,

$d(M,C) = \sqrt{(x - a)^2 + y^2}$,

and $d(P,C) = 2a$.

By employing the Pythagorean Theorem in $\triangle PCM$ we obtain

$$(x + a)^2 + y^2 + (x - a)^2 + y^2 = 4a^2 ,$$

$$x^2 + 2ax + a^2 + y^2 + x^2 - 2ax + a^2 + y^2 = 4a^2 ,$$

$$2x^2 + 2y^2 = 2a^2 ,$$

or

$$x^2 + y^2 = a^2 .$$

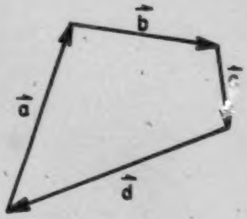
We recognize this as an equation of the circle of radius a which has its center at the origin. However, the entire circle is not the locus in the case we have depicted. The locus is the arc of this circle which is contained in or on the fixed circle. This is the case for which the radius, r, of the fixed circle is less than $2a$; the point P is exterior to the fixed circle. If $r = 2a$, P is on the fixed circle; if $r > 2a$, P is inside the fixed circle. In both of these latter two cases, the entire circle $x^2 + y^2 = a^2$ is the locus.

4-3. Proofs Using Vectors.

The purpose of this section is to show another method of proving geometric porpositions. It is inappropriate to say that one method is superior to another. For a particular problem, one method may be simpler than another method, but the point here is to increase the diversity of available methods. Using vectors may be an approach which, though new to many students can be of considerable interest to them. If the teacher (or any student) wishes to pursue this topic of vectors applied to geometry, he may consult Elementary Vector Geometry by Seymour Schuster.

147 A reference to the discussion of Figure 3-8 in Chapter 3 may help some students to understand the vector addition performed in Example 1. This example is Exercise 13 of the preceding set.

An application of vector addition which may interest some students involves the sum around a closed region. For example, $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$. One of Kirchhoff's Laws, which is widely used in dealing with electrical circuits, states that the sum of the potential (voltage) drops around a closed circuit is zero.



147 The students should discover that altering the directions of any of the vectors in Figure 4-8 will not essentially change the proof--only some details will be modified. The students may encounter some difficulty, however, if they are careless in the way they label the vectors. For example, since E is the midpoint of $\overline{AD}$ and we chose $\vec{a}$ to designate the vector from A to E, the vector from E to D is also labeled $\vec{a}$. But if we used the vector from D to E, it would be labeled $-\vec{a}$.

148 Example 2 is Exercise 10 of Exercises 4-2. We have suggested to the student that he copy Figure 4-9. We should like to emphasize this suggestion. We think this will help the student to see that the choice of an origin is completely arbitrary, and the drawing of the origin-vectors as the proof proceeds may aid in visualizing the steps of the proof.

149 Example 3 is Exercise 16 of Exercises 4-2. Note that a particular choice of origin (aided by a prior knowledge of the result) greatly simplifies the proof.

In solving any sort of problem it is difficult in general to tell beforehand what will "work" and what will not. This is true of the more complicated exercises where a particular choice of the roigin may give simpler calculations than occur with another choice. In general, an origin should be selected which allows the hypothesis to be expressed simply. It should also be chosen so that the number of independent vectors needed is as small as possible. Apart from this, experience gained from trial and error is a valuable help. If calculations bog down with one choice, perhaps another choice should be made. However, some propositions simply do not possess short, elegant proofs.

151 The centroid of an area or a volume can be defined in mathematical terms using integral calculus. The center of gravity of a thin uniform sheet or of a uniform mass is the centroid of the corresponding mathematical area or volume.

Physically, the center of gravity of an object will always lie on a vertical line through a point of suspension of the object. Thus the center of gravity of a triangular object can also be determined experimentally by suspending it from 2 different points, say 2 vertices, and then determining where the lines of suspension intersect.

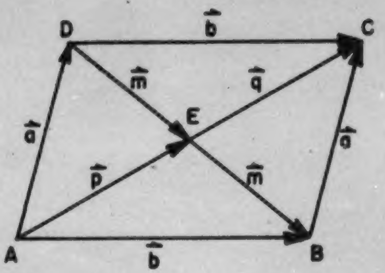
151 There may be some mystery surrounding the choice of unit vectors in Example 4. Of course, we always can say, "It works!" But we can give a more sound justification. The fact that we need an angle bisector could lead someone to think of the diagonals of a rhombus, and the congruent sides of a rhombus could lead someone to think of unit vectors. Students (and teachers) should not be discouraged if they do not think of things like this; years of experience and/or a little luck play a large part in these activities.

153 Exercises 5 and 6 of Section 4-3 are the same theorems used in Examples 3 and 1 of Section 4-2. These may be assigned for purposes of comparing the two methods of proof.

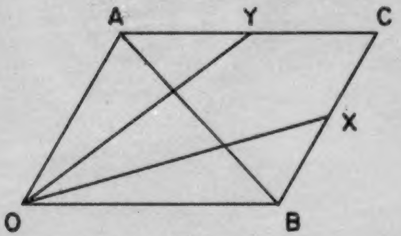
Exercises 4-3

(Note: Formal proofs are not presented here. We merely indicate the essentials of one possible solution for each problem.)

1. Let E be the midpoint of $\overline{DB}$. We have $\vec{p} = \vec{a} + \vec{m}$ and $\vec{q} = \vec{m} + \vec{a}$; therefore, $\vec{p} = \vec{q}$ and point E bisects $\overline{AC}$.



2.



Consider the diagram at the left.

$\overline{AY} \cong \overline{YC}$ $\quad$ $\overline{CX} \cong \overline{XB}$

We wish to show that $\overline{OY}$ and $\overline{OX}$ trisect $\overline{AB}$, and that $\overline{AB}$ passes through points of trisection of $\overline{OY}$ and $\overline{OX}$.

Any point on $\overline{AB}$ can be represented by $z\vec{A} + (1 - z)\vec{B}$, $0 \leq z \leq 1$.

Any point on $\overline{OY}$ can be represented by $y\vec{Y}$, $0 \leq y \leq 1$.

Any point on $\overline{OX}$ can be represented by $x\vec{X}$, $0 \leq x \leq 1$.

We wish to find values of x and z such that $z\vec{A} + (1 - z)\vec{B} = x\vec{X}$.

But we also know $\vec{X} = \frac{1}{2}(\vec{C} + \vec{B})$ and $\vec{C} = \vec{A} + \vec{B}$

so we want $z\vec{A} + (1 - z)\vec{B} = \frac{1}{2}x(\vec{A} + \vec{B} + \vec{B})$

$$z\vec{A} + (1 - z)\vec{B} = \frac{1}{2}x\vec{A} + x\vec{B}$$

so we find $z = \frac{1}{3}$ $\quad$ $x = \frac{2}{3}$

Thus the intersection is at $\frac{1}{3}\vec{A} + \frac{2}{3}\vec{B} = \frac{2}{3}\vec{X}$

We find by similar computations that $\overline{AB}$ intersects $\overline{OY}$ at $\frac{2}{3}\vec{A} + \frac{1}{3}\vec{B} = \frac{2}{3}\vec{Y}$

This means $\overline{OY}$ and $\overline{OX}$ trisect $\overline{AB}$ and also that $\overline{AB}$ passes through points of trisection of $\overline{OX}$ and $\overline{OY}$.

3. Using A as the origin, we have

$$\vec{P} = \frac{1}{2}(\vec{B} + \vec{C}),$$

$$\vec{Q} = \frac{1}{2}\vec{C},$$

$$\vec{R} = \frac{1}{2}\vec{B}.$$

The intersection of medians $\overline{BQ}$ and $\overline{CR}$ can be located by finding the values of x and y which solve

$$x\vec{B} + (1 - x)\vec{Q} = y\vec{C} + (1 - y)\vec{R} .$$

Substituting, we obtain

$$x\vec{B} + \frac{1}{2}\vec{C} - \frac{1}{2}x\vec{C} = y\vec{C} + \frac{1}{2}\vec{B} - \frac{1}{2}y\vec{B}.$$

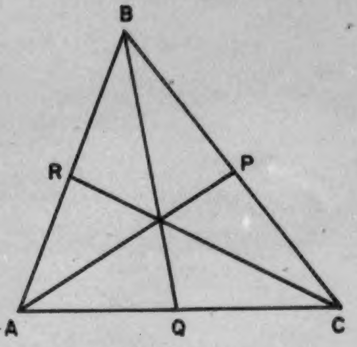


Figure 4-12

Equating corresponding coefficients, we have

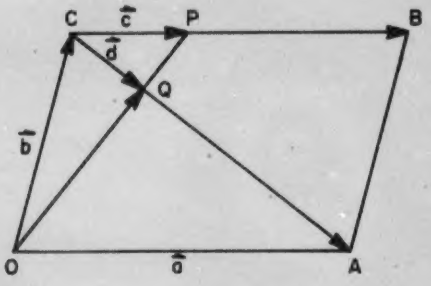
$$x = \frac{1}{2}(1 - y) \text{ and } y = \frac{1}{2}(1 - x),$$

from which we obtain $x = y = \frac{1}{3}$.

This tells us that the intersection of $\overline{BQ}$ and $\overline{CR}$ is $\frac{1}{3}(\vec{B} + \vec{C})$, which is a trisection point of each of these medians. A trisection point of $\overline{AP}$ is

$$\frac{2}{3}\vec{P} = \frac{2}{3}\cdot\frac{1}{2}(\vec{B} + \vec{C}) = \frac{1}{3}(\vec{B} + \vec{C}) .$$

4. Since $\frac{d(C,P)}{d(C,B)} = \frac{1}{r}$, the vector from C to P is $\vec{c} = \frac{1}{r}\vec{a}$. The vector from C to A is $(\vec{a} - \vec{b})$, and we wish to find $n(\vec{a} - \vec{b}) = \vec{d}$, the scalar multiple of it. The vector from O to Q may be expressed as $(\vec{b} + \vec{d})$ or as a scalar multiple of the vector



$\vec{p}$. We therefore have

$$\vec{b} + \vec{d} = m(\vec{b} + \vec{c}),$$

$$\vec{b} + n(\vec{a} - \vec{b}) = m(\vec{b} + \frac{1}{r}\vec{a}),$$

$$\vec{b} + n\vec{a} - n\vec{b} = m\vec{b} + \frac{m}{r}\vec{a}.$$

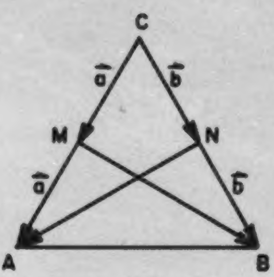
Equating corresponding coefficients gives us

$$n = \frac{m}{r} \quad \text{and} \quad m = (1 - n);$$

for these equations we find $n = \frac{1}{r + 1}$. Therefore,

$$\vec{d} = \frac{1}{r + 1}(\vec{a} - \vec{b}), \text{ and } \frac{d(C,Q)}{d(C,A)} = \frac{1}{r + 1} .$$

5. From the diagram we see that the vector from N to A is $2\vec{a} - \vec{b}$ and the vector from M to B is $2\vec{b} - \vec{a}$. Since $d(N,A) = d(M,B)$, we have $|2\vec{a} - \vec{b}| = |2\vec{b} - \vec{a}|$. Using the Law of Cosines, we may write this as



$$\sqrt{4|\vec{a}|^2 + |\vec{b}|^2 + 4\vec{a}\cdot\vec{b}} = \sqrt{4|\vec{b}|^2 + |\vec{a}|^2 + 4\vec{b}\cdot\vec{a}} .$$

This equation simplifies to

$$4|\vec{a}|^2 + |\vec{b}|^2 = 4|\vec{b}|^2 + |\vec{a}|^2 ,$$

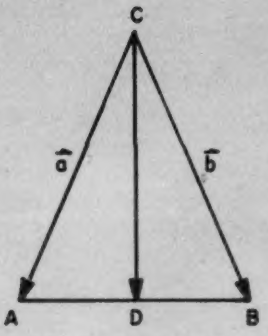
or

$$3|\vec{a}|^2 = 3|\vec{b}|^2.$$

From this we see that $2|\vec{a}| = 2|\vec{b}|$, and $\triangle ABC$ is isosceles.

This vector proof of Example 3, Section 4-2, is somewhat artificial because of the use of the Law of Cosines. It may be profitable for the students to compare this proof with the rectangular coordinate and synthetic proofs appearing in Section 4-2 of this commentary. It can be noted that applying vectors to equal lengths may become awkward if the vectors are not parallel.

6. The vector from C to D may be expressed as $\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$, and the vector from A to B may be expressed as $\vec{b} - \vec{a}$. The product of these two vectors is



$$(b - a)\cdot\left(\frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}\right)$$

$$= \frac{1}{2}\vec{a}\cdot\vec{b} - \frac{1}{2}\vec{a}\cdot\vec{a} + \frac{1}{2}\vec{b}\cdot\vec{b} - \frac{1}{2}\vec{a}\cdot\vec{b}$$

$$= \frac{1}{2}\left(|\vec{b}|^2 - |\vec{a}|^2\right).$$

Since the isoceles triangle has $|\vec{a}| = |\vec{b}|$, the vector product is zero, and $\overline{CD} \perp \overline{AB}$.

7. Let ABCD be a quadrilateral; i.e., A, B, C, D are distinct.

$\vec{M} = \frac{1}{2}(\vec{A} + \vec{B})$ $\qquad$ $\vec{N} = \frac{1}{2}(\vec{B} + \vec{C})$

$\vec{P} = \frac{1}{2}(\vec{C} + \vec{D})$ $\qquad$ $\vec{Q} = \frac{1}{2}(\vec{D} + \vec{A})$

M, N, P, Q are the midpoints of the sides.

We wish to show $\overline{MP}$ bisects $\overline{NQ}$.

Points of $\overline{MP}$: $x\vec{M} + (1 - x)\vec{P}$ $\quad 0 \le x \le 1$

Points of $\overline{NQ}$: $y\vec{N} + (1 - y)\vec{Q}$

Intersection requires that

$$x\vec{M} + (1 - x)\vec{P} = y\vec{N} + (1 - y)\vec{Q}$$

$$x(\tfrac{1}{2}\vec{A} + \tfrac{1}{2}\vec{B}) + (1 - x)(\tfrac{1}{2}\vec{C} + \tfrac{1}{2}\vec{D}) = y(\tfrac{1}{2}\vec{B} + \tfrac{1}{2}\vec{C}) + (1 - y)(\tfrac{1}{2}\vec{D} + \tfrac{1}{2}\vec{A})$$

so $\frac{1}{2}x = \frac{1}{2}(1 - y)$ and $\frac{1}{2}(1 - x) = \frac{1}{2}y$

hence $x = y = \frac{1}{2}$.

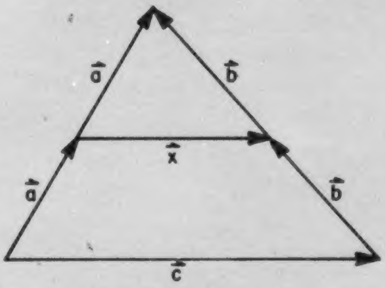
Thus $\overline{MP}$ intersects $\overline{NG}$ in a point which bisects both.

8. $\vec{x} = -\vec{a} + \vec{c} + \vec{b}$;

$\vec{x} = \vec{a} - \vec{b}$.

Adding, we have

$2\vec{x} = \vec{c}$, or $\vec{x} = \frac{1}{2}\vec{c}$.

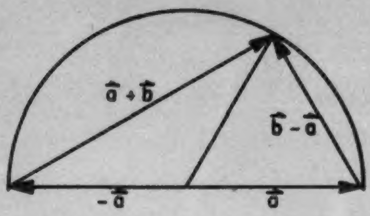


9. $(\vec{a} + \vec{b}) \cdot (\vec{b} - \vec{a})$

$= \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b}$

$= |\vec{b}|^2 - |\vec{a}|^2$.

Since $|\vec{a}| = |\vec{b}|$, $|\vec{b}|^2 - |\vec{a}|^2 = 0$.

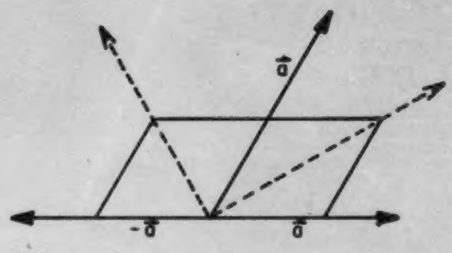

$\vec{a} + \vec{b}$
$\vec{b} - \vec{a}$
$-\vec{a}$
$\vec{a}$

10. As in Example 4, we use unit vectors to express the angle bisectors. Then, taking the vector product, we obtain

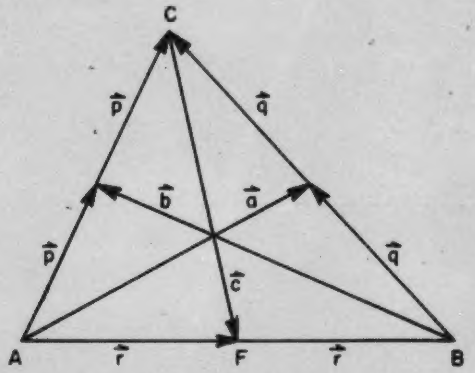
$$\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} \cdot \frac{\vec{b}}{|\vec{b}|} - \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} + \frac{\vec{b} \cdot \vec{b}}{|\vec{b}|^2} - \frac{\vec{a} \cdot \vec{a}}{|\vec{a}|^2} - \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{|\vec{b}|^2}{|\vec{b}|^2} - \frac{|\vec{a}|^2}{|\vec{a}|^2} = 0 .$$


$\vec{a}$
$-\vec{a}$
$\vec{a}$

11.


C
$\vec{p}$
$\vec{q}$
$\vec{b}$
$\vec{a}$
$\vec{p}$
$\vec{q}$
$\vec{c}$
A
$\vec{r}$
F
$\vec{r}$
B

$$\begin{array}{lll}
\vec{a} = 2\vec{r} + \vec{q} & \vec{b} = -2\vec{r} + \vec{p} & \vec{c} = -2\vec{p} + \vec{r} \\
\vec{a} = 2\vec{p} - \vec{q} & \vec{b} = 2\vec{q} - \vec{p} & \vec{c} = -2\vec{q} - \vec{r} \\
2\vec{a} = 2\vec{r} + 2\vec{p} & 2\vec{b} = 2\vec{q} - 2\vec{r} & 2\vec{c} = -2\vec{p} - 2\vec{q}
\end{array}$$

$$\vec{a} + \vec{b} + \vec{c} = \vec{r} + \vec{p} + \vec{q} - \vec{r} - \vec{p} - \vec{q} = \vec{0} .$$

4-4. Proofs Using Polar Coordinates.

Polar coordinates are not particularly adapted for proving theorems of the type we have been discussing. The beauty and usefulness of this form will be more apparent in later chapters. Exercises using polar representation are, therefore, deferred. We have included two examples to illustrate the possibilities for polar coordinates at this point of our progress and to set the stage for the next section.

4-5. Choice of Method of Proof.

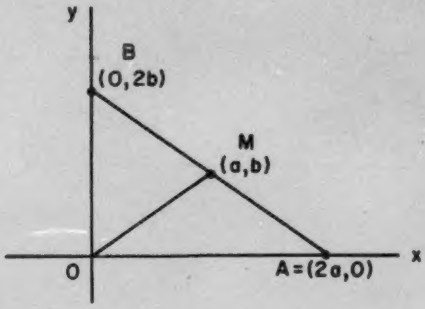
This section, which contains rather specific directions for problem solving, should be carefully read and discussed. Most of the Review Exercises which follow may be used to give the students experience in choosing and following through with some particular method. The solutions we present are merely the ones which occurred to us; they are not put forth as the only ones available or even the best of the many possibilities. As was said before, any mathematically sound presentation should be acceptable.

Review Exercises

1. $d(O,M) = \sqrt{a^2 + b^2}$.

$$d(A,M) = d(B,M) = \sqrt{(2a - a)^2 + b^2}$$

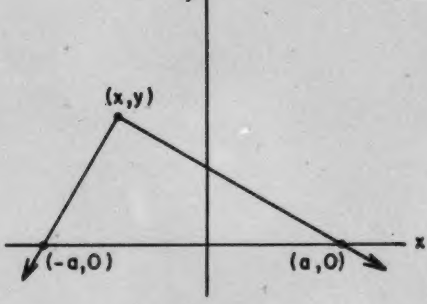
$$= \sqrt{a^2 + b^2} .$$



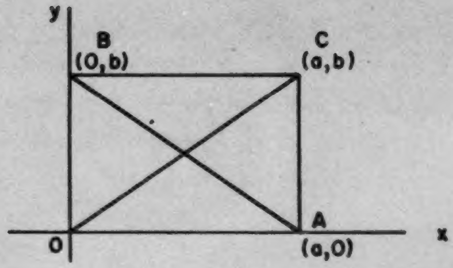
2. Let the fixed points be on the x-axis, as indicated in the figure. By multiplying the slopes of the sides of the angle we have

$$\frac{y}{x - a} \cdot \frac{y}{x + a} = -1 ,$$

$y^2 = -x^2 + a^2$, or $x^2 + y^2 = a^2$.

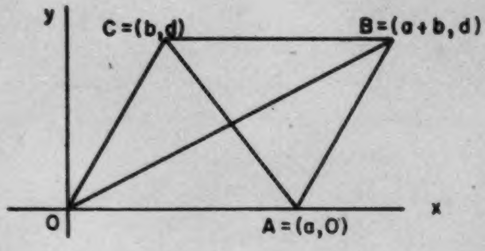


3.


y
B
(0,b)
C
(a,b)
A
0
(a,0)
x

$d(O,C) = \sqrt{a^2 + b^2}$ and $d(A,B) = \sqrt{a^2 + b^2}$

4.


y
C=(b,d)
B=(a+b,d)
0
A=(a,0)
x

The coordinates of B are $(a + b,d)$.

$$\big(d(O,A)\big)^2 + \big(d(A,B)\big)^2 + \big(d(B,C)\big)^2 + \big(d(C,O)\big)^2$$

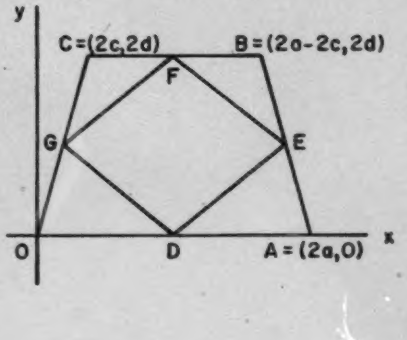
$$= a^2 + (b^2 + d^2) + a^2 + (b^2 + d^2)$$

$$= 2(a^2 + b^2 + d^2)$$

$$\big(d(O,B)\big)^2 + \big(d(A,C)\big)^2 = \big((a + b)^2 + d^2\big) + \big((a - b)^2 + d^2\big)$$

$$= 2(a^2 + b^2 + d^2).$$

5. $D = (a,0)$; $E = (2a - c,d)$;
$F = (a,2d)$; $G = (c,d)$.
From Exercise 10 of Exercises 4-2, we know that DEFG is a parallelogram; from Exercise 9 of Exercises 4-2, we know that DEFG is a rhombus if $\overline{DF} \perp \overline{GE}$. It is evident from the coordinates of the midpoints that $\overline{DF}$ is vertical and $\overline{GE}$ is horizontal.

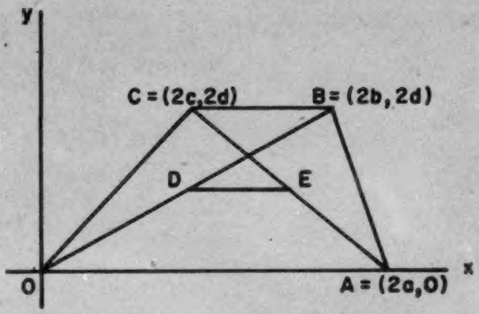

y
C=(2c,2d)
B=(2a-2c,2d)
F
G
E
0
D
A=(2a,0)
x

6. $D = (b,d)$; $E = (a + c,d)$.
It is evident from the coordinates that $\overline{OA}$, $\overline{BC}$, and $\overline{DE}$ are horizontal and, hence, parallel.

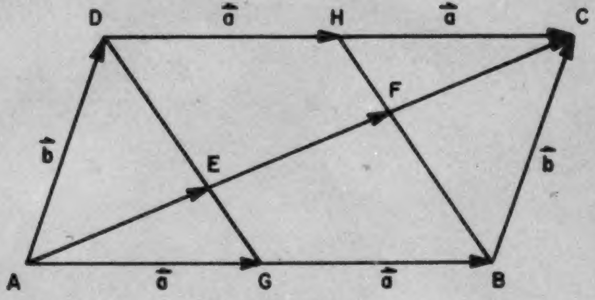
$d(O,A) - d(B,C) = 2a - (2b - 2c)$

$= 2(a - b + c)$.

$d(D,E) = a + c - b$.



7.



The vector from D to G is $\vec{a} - \vec{b}$; the vector from H to B is $2\vec{a} - \vec{b} - \vec{a} = \vec{a} - \vec{b}$; hence, $\overleftrightarrow{DG} \parallel \overleftrightarrow{HB}$. The vector from A to E may be represented by $x\vec{a} + (1 - x)\vec{b}$ or by $y(2\vec{a} + \vec{b})$. Setting these equal we have

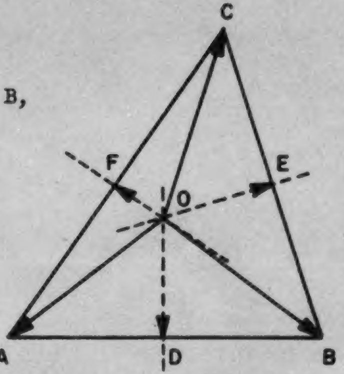
$$x\vec{a} + (1 - x)\vec{b} = 2y\vec{a} + y\vec{b} .$$

Equating coefficients results in $x = 2y$, $y = 1 - x$. Solving these equations together gives us $y = \frac{1}{3}$. The vector from A to F may be represented by $x(2\vec{a}) + (1 - x)(\vec{a} + \vec{b})$ or by $y(2\vec{a} + \vec{b})$. Equating these, we obtain $y = \frac{2}{3}$.

8. Let D, E, and F be the midpoints of the sides, and let the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ intersect at the origin. Since $\vec{D}$ is perpendicular to the vector from A to B, $\frac{1}{2}(\vec{A} + \vec{B})\cdot(\vec{B} - \vec{A}) = 0$, or $\frac{1}{2}(\vec{B}\cdot\vec{B} - \vec{A}\cdot\vec{A}) = 0$; therefore $|\vec{B}|^2 = |\vec{A}|^2$. Similarly, $|\vec{A}|^2 = |\vec{C}|^2$. Since $\vec{F} = \frac{1}{2}(\vec{A} + \vec{C})$, $\frac{1}{2}(\vec{A} + \vec{C})\cdot(\vec{A} - \vec{C})$

$$= \frac{1}{2}(\vec{A}\cdot\vec{A} - \vec{C}\cdot\vec{C})$$

$$= \frac{1}{2}(|\vec{A}|^2 - |\vec{C}|^2).$$

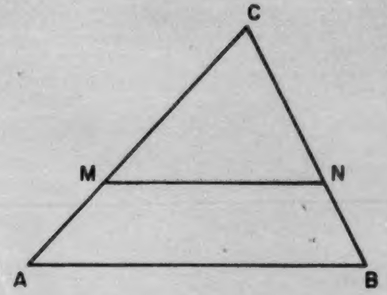


But since $|\vec{A}|^2 = |\vec{C}|^2$, $\frac{1}{2}(|\vec{A}|^2 - |\vec{C}|^2) = 0$, and $\vec{F}$ is perpendicular to the vector from C to A. Consequently the perpendicular bisector of $\overline{AC}$ intersects the other two perpendicular bisectors at O.

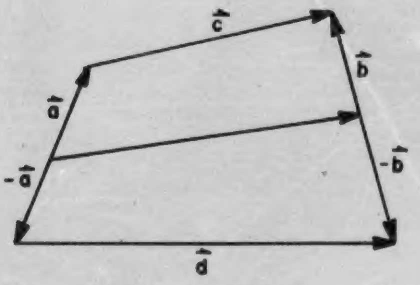
9. Let M and N divide $\overline{AC}$ and $\overline{BC}$ in the same ratio, r. Then, $\vec{M} - \vec{N}$

$= (r\vec{A} + (1 - r)\vec{C}) - (r\vec{B} + (1 - r)\vec{C})$

$= r\vec{A} - r\vec{B} = r(\vec{A} - \vec{B})$.



10.

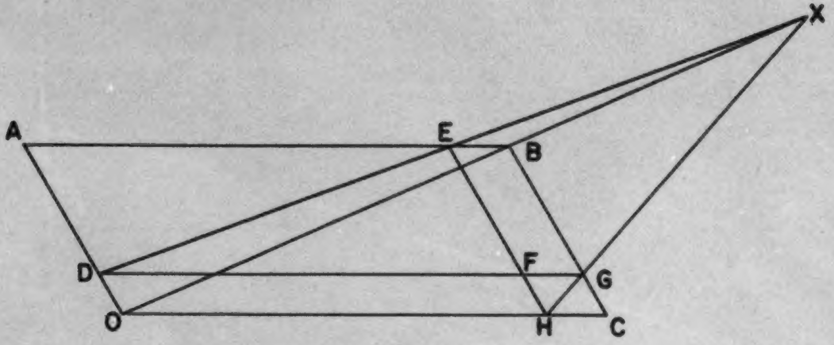


$\vec{x} = \vec{a} + \vec{c} - \vec{b}$,

$\vec{x} = -\vec{a} + \vec{d} - (-\vec{b})$.

Adding, we obtain $2\vec{x} = \vec{c} + \vec{d}$, or $\vec{x} = \frac{1}{2}(\vec{c} + \vec{d})$.

11.



We are given parallelograms ABCO, AEFD, FGCH.

Define numbers d, h such that $\vec{D} = d\vec{A}$; $\vec{H} = h\vec{C}$.

We will express everything in terms of d, h, $\vec{A}$, $\vec{C}$ and assume all points are distinct.

The line through $\overline{DE}$ contains points $x\vec{D} + (1 - x)\vec{E}$

or $x(d\vec{A}) + (1 - x)(\vec{A} + h\vec{C})$.

The line through $\overline{HG}$ contains points $y\vec{H} + (1 - y)\vec{G}$

or $yh\vec{C} + (1 - y)(\vec{C} + d\vec{A})$.

For these two lines to intersect, we must have

$$(xd + 1 - x)\vec{A} + (1 - x)h\vec{C} = (1 - y)d\vec{A} + (yh + 1 - y)\vec{C} .$$

Thus we must have

$$yh + 1 - y = h - xh$$

$$xd + 1 - x = d - yd.$$

Solving this system we get, under condition that $h \neq 1 - d$,

$$y = \frac{d - 1}{h + d - 1} \qquad x = \frac{h - 1}{h + d - 1}$$

which puts the intersection at X such that

$$\vec{X} = \frac{hd}{h + d - 1} \vec{A} + \frac{hd}{h + d - 1} \vec{C} = \frac{hd}{h + d - 1} (\vec{A} + \vec{C}) .$$

From this we see immediately that X lies on the line containing $\overline{OB}$ since $\vec{A} + \vec{C} = \vec{B}$.

The restriction $h \neq 1 - d$ arises because in the case $h = 1 - d$, we get $\frac{|\vec{A}|}{|\vec{C}|} = h = 1 - d = \frac{1 - |\vec{D}|}{|\vec{A}|}$ which makes the parallelograms similar and the diagonals parallel.

12. Since $d(A,P) = d(Q,B)$, $\overrightarrow{P}$ can be represented by $\overrightarrow{A} + \overrightarrow{P}(\overrightarrow{B} - \overrightarrow{A})$ and $\overrightarrow{Q}$ by $\overrightarrow{B} + p(\overrightarrow{A} - \overrightarrow{B})$.

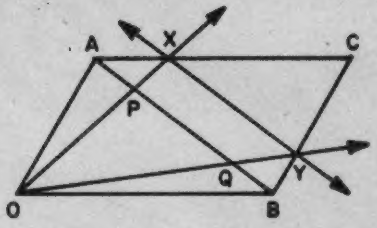
$\overrightarrow{x} = \overrightarrow{A} + k(\overrightarrow{C} - \overrightarrow{A})$ and $\overrightarrow{x} = q\overrightarrow{P}$

so that

$\overrightarrow{A} + k(\overrightarrow{C} - \overrightarrow{A}) = q\overrightarrow{P}$,

$\overrightarrow{A} + k(\overrightarrow{A} + \overrightarrow{B} - \overrightarrow{A}) = q\left(\overrightarrow{A} + p(\overrightarrow{B} - \overrightarrow{A})\right)$,

$\overrightarrow{A} + k\overrightarrow{B} = q(1 - p)\overrightarrow{A} + qp\overrightarrow{B}$.



Equating coefficients, we have

$1 = q(1 - p)$ and $k = qp$;

therefore,

$k = \frac{p}{1 - p}$ and $\overrightarrow{x} = \overrightarrow{A} + \frac{p}{1 - p}\overrightarrow{B}$.

A similar argument gives us $\overline{Y} = \overline{B} + \frac{p}{1 - p}\overrightarrow{A}$.

Thus,

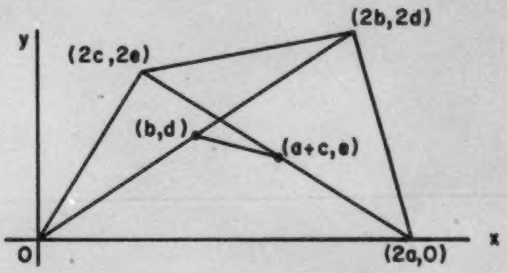
$$\overrightarrow{X} - \overrightarrow{Y} = \overrightarrow{A} + \frac{p}{1 - p}\overrightarrow{B} - \overrightarrow{B} - \frac{p}{1 - p}\overrightarrow{A}$$

$$= (1 - \frac{p}{1 - p})(\overrightarrow{A} - \overrightarrow{B}) ;$$

hence,

$$\overleftrightarrow{XY} \| \overleftrightarrow{AB} .$$

13. The sum of the squares of the lengths of the four sides is



$$(2a)^2 + (2b - 2a)^2 + (2d)^2 + (2b - 2c)^2 + (2d - 2e)^2 + (2c)^2 + (2e)^2$$

$$= 8a^2 + 8b^2 + 8c^2 + 8d^2 + 8e^2 - 8ab - 8bc - 8de .$$

The sum of the squares of the lengths of the diagonals is

$$(2b)^2 + (2d)^2 + (2c - 2a)^2 + (2e)^2$$

$$= 4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 - 8ac .$$

Subtracting these sums, we obtain,

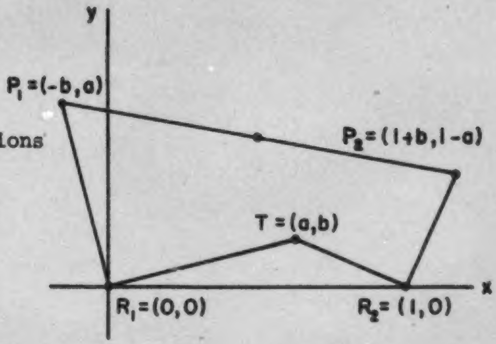
$$4a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 + 8ac - 8ab - 8bc - 8de$$

$$= 4(a^2 + b^2 + c^2 + d^2 + e^2 + 2ac - 2ab - 2bc - 2de) .$$

The square of the length of the line segment joining the midpoints of the diagonals is

$$(a + c - b)^2 + (e - d)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + 2ac - 2ab - 2bc - 2de .$$

14. We select coordinates for the two rocks and the tree as shown in the diagram. After marching the required distances and directions from the rocks, the positions P_1 and P_2 are located. The midpoint of $\overline{P_1P_2}$ is $(\frac{1}{2}, \frac{1}{2})$; therefore, the buried treasure is located at the center of the square whose side is determined by the two rocks. (The location of the tree is unimportant.)



Chapter 5

GRAPHS AND THEIR EQUATIONS

The material of this chapter starts with familiar content including much that has been encountered in earlier courses. The treatment is broader and deeper here than before. It is broader because we now have analytic representations in rectangular, polar, vector, and parametric forms. It is deeper because we take account of some troublesome details and special cases that are not adequately treated on a more elementary level. The work is consequently a bit more difficult, but also more rewarding.

We call particular attention to the treatment of related polar equations, and of paths, as distinguished from curves. Neither treatment is met in a traditional first course in analytic geometry, but we feel that they illuminate some significant mathematical content that is appropriate to this work.

There are many exercises, but, as has been mentioned before in this book, they need not all be assigned. We particularly urge the teacher to exploit a viewpoint we recommended to students. Stress the dynamic aspect of the relationship between geometry and algebra. Some appropriate questions here are, "What would be the effect in the graph if we changed this 5 to -5?"; "What change would we have to make in the equation if we wanted to raise the graph 3 units?; if we wanted a larger circle?; if we wanted only the portion in the first quadrant?"; "What kind of graphs would we get if we replaced this 6 by a variable m, and then took larger and larger values of m?".

Exercises 5-2

1. $y = 3$
2. $x = -5$
3. $y = x$ and $y = -x$; or $x^2 = y^2$
4. $y = \pm 2x$; or $y^2 = 4x^2$
5. $r = a$; or $x^2 + y^2 = a^2$
6. $(x - 3)^2 + (y + 2)^2 = a^2$

7. $x = -1$

8. $3x - 7y - 14 = 0$

9. $\sqrt{5}|x + y - 2| = \sqrt{2}|x + 2y + 2|$; or

$(\sqrt{5} + \sqrt{2})x + (\sqrt{5} + 2\sqrt{2})y - 2\sqrt{5} + 2\sqrt{2} = 0$, and

$(\sqrt{5} - \sqrt{2})x + (\sqrt{5} - 2\sqrt{2})y - 2\sqrt{5} - 2\sqrt{2} = 0$.

10. $y^2 = 8x$

11. If $P = (x,y)$ is a point of the locus, then the distance from P to the line is $\frac{|2x + y + 2|}{\sqrt{5}}$, and from P to the point $(2,0)$ is $\sqrt{(x - 2)^2 + (y + 1)^2}$. The statement of equality of these two distances yields our equation: $x^2 - 4xy + 4y^2 - 28x + 6y + 21 = 0$.

12. $9x^2 + 25y^2 = 225$

13. $7x^2 - 9y^2 = 63$

14. $18x^2 + 48xy + 7y^2 - 156x - 68y + 142 = 0$

15. $5x - 6y + 17 = 0$

16. $\left((x - x_1)^2 + (y - y_1)^2\right)\left((x - x_2)^2 + (y - y_2)^2\right) = k^2$, $k > 0$

17. $-3 \le y \le 3$

18. $x^2 + y^2 \ge 25$

19. $-1 \le x \le 1$

20. $(x - 1)^2 + (y - 3)^2 \le 2^2$, or $x^2 + y^2 - 2x - 6y + 6 \le 0$

21. $y > \frac{5}{2}$

22. $x^2 + 8y \ge 16$

23. $y^2 \le 100 - 20x$

24. $-6 < x < 6$; or $|x| < 6$

25. $x^2 + y^2 < (8.08)^2$; or $x^2 + y^2 < 65.2864$

5-3. Parametric Representation.

The content and treatment of the material in this section are closely related to the physical and scientific applications that pupils will meet in other classes and in later work. Science teachers in the school should be shown this section, and their cooperation solicited in devising laboratory experiments along the lines suggested.

Exercises 5-3

1.

t	0	1	2	3	4	5	6	7	8	9	10
x	0	2	8	18	32	50	72	98	128	162	200
y	0	3	12	27	48	75	108	147	192	243	300

2.

t	0	1	2	3	4	5	6	7	8	9	10
x	0	176	352	528	704	880	1056	1232	1408	1584	1760
y	0	16	64	144	256	400	576	784	1024	1296	1600

3. $\begin{cases} x = 5t \,, \\ y = 2 \,. \end{cases}$

4. $\begin{cases} x = -6 \,, \\ y = 2t \,. \end{cases}$

5. $\begin{cases} x = .3t \,, \\ y = .4t \,. \end{cases}$

6. $\begin{cases} x = -6 + \frac{7}{5}t \,, \\ y = 1 + \frac{24}{5}t \,. \end{cases}$

7. Eliminating the parameter gives $y = x^2$. With the usual placement of the axes this means that the point starts from rest at the origin and moves steadily to the right as it moves more and more rapidly upward. Its path is along a parabola whose vertex is at the origin and which is concave upward. Since we assume $t > 0$, the point travels on only the right half of the parabola. 25.9 units.

8. For the line $4x - 3y + 2 = 0$ we have direction numbers for the normal, $(4,-3)$. Therefore we may take direction numbers for the line as either $(3,4)$, or $(-3,-4)$. Since no sense of direction along the line is specified we must consider both. If we use direction consines then the displacement along the line will be one unit for each unit interval of the parameter t . Since the given rate is 10 units per second we must now take direction numbers ten times the direction cosines, i.e., $\left(10(\frac{3}{5}), 10(\frac{4}{5})\right)$. Since the point goes through $(1,2)$ at the time when $t = 3$, the elapsed time after that is indicated by $t - 3$. We have, in the first case, therefore,

$$\begin{cases} x = 1 + 6(t - 3) , \\ y = 2 + 8(t - 3) ; \end{cases} \quad \text{or} \quad \begin{cases} x = -17 + 6t , \\ y = -22 + 8t ; \end{cases}$$

and in the second case,
$$\begin{cases} x = 19 - 6t , \\ y = 26 - 8t . \end{cases}$$

In the first case, when $t = 0$ the position is $(-17,-22)$, and when $t = 10$ the position is $(43,58)$. In the second case, when $t = 0$ the position is $(19,26)$, and when $t = 10$ the position is $(-41,-54)$.

9. Refer to the solution of (8) above.

$$\begin{cases} x = 3 + \frac{15}{\sqrt{13}}t , \\ y = 0 - \frac{10}{\sqrt{13}}t ; \end{cases} \quad \text{or} \quad \begin{cases} x = 3 - \frac{15}{\sqrt{13}}t , \\ y = 0 + \frac{10}{\sqrt{13}}t . \end{cases}$$

10. Assume $t_1 > t_0$. Direction numbers for the line are $(c - a , d - b)$, and direction cosines $\frac{c - a}{\sqrt{(c - a)^2 + (d - b)^2}} , \frac{d - b}{\sqrt{(c - a)^2 + (d - b)^2}}$.

The velocity of the point along the line is $\frac{\sqrt{(c - a)^2 + (d - b)^2}}{t_1 - t_0}$,

and this is the factor by which we must multiply the direction cosines so that unit intervals of the parameter t correspond properly to displacements along the line. Since the point goes through (a,b) at time t_0 we indicate with our parameter t the elapsed time since then, $t - t_0$. Therefore we have the parametric equations:

$$\begin{cases} x = a + \dfrac{\sqrt{(c-a)^2+(d-b)^2}}{t_1 - t_0} \; \dfrac{c-a}{\sqrt{(c-a)^2+(d-b)^2}}(t - t_0) \, , \\[2ex] y = b + \dfrac{\sqrt{(c-a)^2+(d-b)^2}}{t_1 - t_0} \; \dfrac{d-b}{\sqrt{(c-a)^2+(d-b)^2}}(t - t_0) \, . \end{cases}$$

These formidable equations become:

$$\begin{cases} x = a + \dfrac{c-a}{t_1 - t_0}(t - t_0) \, , \\[2ex] y = b + \dfrac{d-b}{t_1 - t_0}(t - t_0) \, . \end{cases}$$

You may easily verify from these equations that when $t = t_0$ the position is (a,b), and when $t = t_1$ the position is (c,d).

11. Assume t in seconds. The point moves from the point $(1,0)$ to the point $(-1,0)$ and back again, making a round trip in 2π seconds. It starts from rest at $(1,0)$, increases its speed until it reaches the origin, then slows down until it comes to rest momentarily at $(-1,0)$, then reverses the process endlessly. Its maximum speed occurs each time at the origin. (By methods of the calculus this maximum speed can be shown to be one unit per second at that instant.) Such motion is called a "simple harmonic motion" and has many physical applications.

t	0	1	2	3	4	5	6	7	8	9	10
x	1	.540	-.418	-.990	-.652	.287	.961	.752	-.150	-.913	-.836

At the end of one minute $t = 60$, and Table II does not give corresponding values for $\cos t$. We use the fact that $\cos t$ is periodic, of period 2π. (These matters will be developed further in the next chapter.)

We express 60 as a multiple of π and a remainder less than π, which we find by dividing 60 by a suitable decimal equivalent of π. Tables I and II are given correct to three significant figures and a careless student may then take 3.14 as a proper equivalent of π. However, any inaccuracy in this approximation will be multiplied by a factor of about 20 and will give us a seriously inaccurate answer.

It is not our intention to enter into an extended discussion of significant figures and accuracy of computation, but in this exercise we caution that we must choose an appropriate approximation of π.

We assume $t = 60 = 60.0000$, and use $\pi \approx 3.1416$ and obtain $60.0000 = 19\pi + .3096$, which we write briefly as $60 = 19\pi + .310$. Therefore $\cos 60 = \cos(19\pi + .310) = -\cos .310 = -.952$.

In the same way we assume t for one hour to equal 3600.0000000, not 3600, and then take the proper approximation, $\pi \approx 3.141593$. Then $3600.0000000 = 1145\pi + 2.876015$, or $3600.0000000 = 1146\pi - .285578$, which we write more briefly as $3600 = 1146\pi - .286$. Thus $\cos 3600 = \cos(1146\pi - .286) = \cos(-.286) = \cos .286 \approx .959$.

You need not belabor the details of approximate computation, but this is a good place to show the need for a proper approximation for π. It is also a good place to show that when we are working with measurements and we add zeros to the dividend in division we are assuming more and more accuracy in its determination. A measurement of 10. inches is less accurate than one of 10.0 inches which is in turn less accurate than a measurement of 10.00 inches. We particularly warn against the error of dividing a 10 inch length into three equal parts and writing the length of one part as 3.3333.... inches!

12. The motion could be that of an object dropped from an altitude of 500 feet, in which case we assume no air resistance, and a value of 16 feet per second per second as the acceleration due to gravity. A value of y represents the altitude, in feet, above the surface of the earth, at corresponding time t, in seconds after the instant of release. The change of sign of y in the interval $t = 5$ to $t = 6$ can be interpreted to mean that the object reaches the surface of the earth in that interval. The negative values of y afterwards would indicate the depth below the surface, if the fall continued down a vertical shaft.

t	0	1	2	3	4	5	6	7	8	9	10
y	500	484	436	356	244	100	-76	-284	-524	-796	-1100

13. (Refer to the solution of Exercise 12) This equation could represent the motion of an object hurled upward at 64 feet per second from an altitude of 120 feet.

t	0	1	2	3	4	5	6	7	8	9	10
y	120	168	184	168	120	40	-72	-216	-392	-600	-840

14. (Refer to the solution of Exercise 11.) This equation could describe a simple harmonic motion with these conditions: The point starts from a position of rest at the origin; moves, in the next $\frac{\pi}{4}$ seconds, to its farthest right position at $(4,0)$ where it halts momentarily and reverses direction to move to its farthest left position at $(-4,0)$, arriving there in an additional $\frac{\pi}{2}$ seconds. It accelerates from $(4,0)$ to the origin where it attains its maximum velocity, then decelerates from the origin to $(-4,0)$, and so on making a round trip in π seconds. Such equations of motions occur in the study of vibrations, and of variations of an alternating current.

t	0	1	2	3	4	5	6	7	8	9	10
x	0	3.636	-3.032	-1.104	3.956	-2.192	-2.124	3.864	-1.184	-2.980	3.668

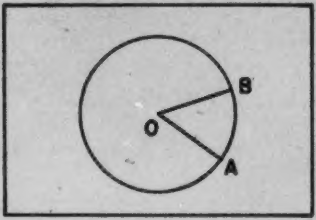
15. (Refer to the solution of Exercise 11.) The point now starts from $(1,0)$ and moves to $(3,0)$ and back, as before, making the round trip in 2π seconds.

16. We must assume they start at the same instant, in which case the variable t has the same interpretation in both equations. Therefore $\cos t = 2 - \cos t$, from which we get $\cos t = 1$ and $t = 0, 2\pi, 4\pi, \ldots$. For these values of t, $x = 1$, therefore the points start together at $(1,0)$, and rendezvous there every 2π seconds thereafter.

5-4. Parametric Equations of the Circle and the Ellipse.

Focal phenomena are familiar enough to physics, but it is interesting to see how the associated mathematical analysis can be used in other situations. Authors in recent publications have applied these concepts in such areas as: epidemiology, to study the spread and control of disease; demography, to study the distributions of groups of people; bacteriology, to study the spread or control of bacterial growth; communication theory, to study the distribution of "information", and so on. We leave these for later years, and concern ourselves now with the simplest and most natural of the applications of parametric equations of the circle, that is, circular paths.

The teacher is urged to make a simple visual aid: The essential features are two movable radii $\overline{OA}$ and $\overline{OB}$ mounted on a panel of suitable size. Two students can then give independent motions to points on the rim of the circle. This model will be particularly useful when you get to problems of "meeting" or "overtaking".



Exercises 5-4

1. $\begin{cases} x = 10 \cos \theta \ , \\ y = 10 \sin \theta \ . \end{cases}$

2. We assume t in seconds. A clockwise rotation means that as t increases from 0, θ decreases from 0, and in this case a rate of 4 rps gives the angular displacement, $-8\pi t$. The equations are

$$\begin{cases} x = 10 \cos(-8\pi t) \ , \\ y = 10 \sin(-8\pi t) \ . \end{cases}$$

3. Consider $x = a \cos(b + \omega t)$. Since the radius is 6 inches, then $a = 6$ and we are committed to inches as the measure of x.

Since the numbers 0 and 60 are assigned to the 12 o'clock position the units of rotation in this problem are intended to be minutes. The angular position of any point on the rim can be given in terms of these m-units, measured from the 12 o'clock position, or in terms of the usual θ, in radian units from the polar axis. Thus the 2 o'clock position can be described by $m = 10$, and also by $\theta = \frac{\pi}{6}$. Since we rotate clockwise at the rate of one rotation in 60 minutes we have ω, the directed rate of angular displacement, equal to 1 m-unit per minute, or $\frac{-\pi}{30}$ radians per minutes.

If in the equation $x = a \cos(b + \omega t)$ we use radian units for b we have $b = \frac{\pi}{2}$, since we start from the 12 o'clock position. Finally, since we are asked for the path during one hour, we take $0 < t < 60$. The result of all this discussion is the following pair of equations:

$$\begin{cases} x = 6 \cos(\frac{\pi}{2} - \frac{\pi}{30}t) \ , \\ y = 6 \sin(\frac{\pi}{2} - \frac{\pi}{30}t) \ , \end{cases} \quad 0 \leq t < 0 \ .$$

t is the time in minutes, x and y are in inches, and the angle is measured as usual in radians, counterclockwise from the polar axis.

4. $\begin{cases} x = 4 + 3\cos\theta , \\ y = 3\sin\theta . \end{cases}$

5. $\begin{cases} x = 4\cos\theta , \\ y = 6 + 4\sin\theta . \end{cases}$

6. $\begin{cases} x = 4 + 3\cos(-\frac{\pi}{2} - 4\pi t), \\ y = 3\sin(-\frac{\pi}{2} - 4\pi t) . \end{cases}$

Note: These equations supply information about the starting position $(-\frac{\pi}{2})$, and the direction and speed of rotation (-4π), but for purposes of computation they may be replaced by the equivalent equations,

$$\begin{cases} x = 4 + \cos(\frac{\pi}{2} + 4\pi t) , \\ y = -3\sin(\frac{\pi}{2} + 45t) . \end{cases}$$

These latter equations show that the path of the point P of exercise 6 is the reflection in the x-axis of the path of the point P' whose equations are

$$\begin{cases} x' = 4 + \cos(\frac{\pi}{2} + 4\pi t) , \\ y' = 3\sin(\frac{\pi}{2} + 4\pi t) . \end{cases}$$

The point P' starts at the highest point of its path and moves counterclockwise, as we should expect the reflected point to do.

7. $\begin{cases} x = 4\cos(\frac{\pi}{2} + 6\pi t) , \\ y = 6 + 4\sin(\frac{\pi}{2} + 6\pi t) . \end{cases}$

8. The point moves around a circle whose center is the origin and whose radius is 4 . The point starts from the 3 o'clock position and moves counterclockwise at the rate of $\frac{1}{2}$ rotation per second.

9. The point moves around a circle whose center is the origin and whose radius is 6 . It starts from the 12 o'clock position and moves clockwise at the rate of $\frac{1}{2}$ rps.

Note: In Solutions 10-16 the paths are all circular, and we shall condense the information which could be written out in full as in (8) and (9) above.

10. Circle; center, origin; $r = 8$; start, 9 o'clock position; direction, clockwise; rate, $\frac{3}{2}$ rps.

11. Circle; center, origin; $r = 10$; start, 6 o'clock position; direction, counterclockwise; rate, 5 rps.

12. Circle; center, $(4,0)$; radius, 1 ; start, 3 o'clock position; direction, counterclockwise; rate, 3 rps.

13. Circle; center, $(0,-3)$; radius, 1 ; start, 3 o'clock position; direction, counterclockwise; rate, 4 rps.

14. Circle; center, $(2,5)$; radius, 1 ; start, 3 o'clock position; direction, counterclockwise; rate, 6 rps.

15. Circle; center, (a,c) ; radius, b ; start, 3 o'clock position; direction, counterclockwise; rate, 1 rps.

16. Circle; center, (p,r) ; radius q ; start, at the angular position $-\alpha$ on the circle; direction, counterclockwise if $n < 0$, no motion at all if $n = 0$; rate, n rps.

17. (a) Circle; center, origin; radius, 6 ; start, 3 o'clock position; direction, counterclockwise; rate, 2 rps.

(b)

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
x	6.00	1.84	-4.88	-4.88	1.84	6.00	1.84	-4.88	-4.88	1.84	6.00
y	0	5.71	3.49	-3.49	-5.71	0	5.71	3.49	-3.49	-5.71	0

(c) $$\begin{cases} x = 6\cos(\frac{\pi}{2} + 4\pi t) , \\ y = 6\sin(\frac{\pi}{2} + 4\pi t) . \end{cases}$$

(d) $$\begin{cases} x = 6\cos(-2\pi t) , \\ y = 6\sin(-2\pi t) . \end{cases}$$

(e) Since the first and third points move in opposite directions, they will meet when the sum of their angular displacements equals their original separation, and, after that, when their additional angular displacements add to an integral multiple of 2π . That is, $2\pi t + 4\pi t = 0$, since they start together, from which $t = 0$, and the points are at $(6,0)$. After that, $2\pi t + 4\pi t = 2\pi , 4\pi , 6\pi$, ... , that is, $t = \frac{1}{3} , \frac{2}{3} , 1 , \frac{4}{3} , \ldots$. The points start together, and meet every $\frac{1}{3}$ second thereafter. The corresponding points are $(6,0)$, $(-3,-5.196)$, $(-3,5.196)$, $(6,0)$, $(-3,-5.196)$, ...

(f) As in the previous part, we add the angular displacements, and find the first meeting point when this sum is equal to their original angular separation: that is, when $2\pi t + 4\pi t = \frac{\pi}{2}$. Thus they meet first when $t = \frac{1}{12}$, at the point $(5.196,-3)$. Then we find, as above, their subsequent meetings take place every $\frac{1}{3}$ second, which should be expected, since the first and second points are traveling at the same rate. The meetings therefore take place when $t = \frac{1}{12}, \frac{5}{12}, \frac{9}{12}, \frac{13}{12}, \ldots$, at $(5.196,-3)$, $(-5.196,-3)$, $(6,0)$, $(5.196,-3)$, ...

18. (a)

$$A: \begin{cases} x = \frac{1}{2\pi}\cos(\frac{7}{6}\pi - \frac{2}{3}\pi t), \\ y = \frac{1}{2\pi}\sin(\frac{7}{6}\pi - \frac{2}{3}\pi t). \end{cases}$$

$$B: \begin{cases} x = \frac{1}{2\pi}\cos(\frac{11}{6}\pi - \frac{1}{2}\pi t), \\ y = \frac{1}{2\pi}\sin(\frac{11}{6}\pi - \frac{1}{2}\pi t). \end{cases}$$

$$C: \begin{cases} x = \frac{1}{2\pi}\cos(\frac{1}{2}\pi + \frac{2}{5}\pi t), \\ y = \frac{1}{2\pi}\sin(\frac{1}{2}\pi + \frac{2}{5}\pi t). \end{cases}$$

(b) A: When $t = 0, 3, 6, 9$, position is $(-\frac{\sqrt{3}}{4\pi}, -\frac{1}{4\pi})$;

When $t = 1, 4, 7, 10$, position is $(0, \frac{1}{2\pi})$;

When $t = 2, 5, 8$, position is $(\frac{\sqrt{3}}{4\pi}, -\frac{1}{4\pi})$.

B: When $t = 0, 4, 8$, position is $(\frac{\sqrt{3}}{4\pi}, -\frac{1}{4\pi})$;

When $t = 1, 5, 9$, position is $(-\frac{1}{4\pi}, -\frac{\sqrt{3}}{4\pi})$;

When $t = 2, 6, 10$, position is $(-\frac{\sqrt{3}}{4\pi}, \frac{1}{4\pi})$;

When $t = 3, 7$, position is $(\frac{1}{4\pi}, \frac{\sqrt{3}}{4\pi})$.

C: When $t = 0, 5, 10$, position is $(0, .159)$;
When $t = 1, 6$, position is $(-.151, .049)$;
When $t = 2, 7$, position is $(-.094, -.129)$;
When $t = 3, 8$, position is $(.094, -.129)$;
When $t = 4, 9$, position is $(.151, .049)$;

(c) By the methods of the solution of Exercise 17 we find:

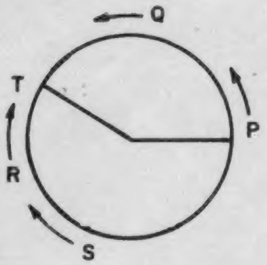
(1) A and C meet when $t = .625$, at $(-.112, .112)$;
(2) B and C meet when $t = 1.480$, at $(-.152, -.046)$;
(3) A and C meet when $t = 2.500$, at $(0, -.159)$;
(4) B and C meet when $t = 3.700$, at $(.159, -.008)$;
(5) A and C meet when $t = 4.375$, at $(.112, .112)$.

(d) By the methods already referred to we find that A and C meet in $\frac{5}{8}$ seconds and every $\frac{15}{8}$ seconds thereafter. That is, their meetings take place at times $t = \frac{5}{8} + \frac{15}{8}p$, where p is a positive integer. In the same way, we find that B and C meet in $\frac{40}{27}$ seconds and every $\frac{20}{9}$ seconds thereafter. That is, the B and C meetings take place when $t = \frac{40}{27} + \frac{20}{9}q$, where q is a positive integer. If A, B, and C are all to meet, there must be a time at which the A, C, and the B, C meetings occur simultaneously. That is, there must be positive integral values of p and q such that $\frac{5}{8} + \frac{15}{8}p = \frac{40}{27} + \frac{20}{9}q$. This equation is equivalent to $81p - 96q = 37$. In this equation, however, the left member is evenly divisible by 3 but the right member is not, therefore there can be no integral values of p and q to satisfy it. Therefore there can be no common meeting of A, B, and C.

19. Since the points move in reflected paths with respect to the y-axis, the second point must start from the position symmetric to A, that is, at $(-\pi, 0)$, where the angular displacement from A is π. Therefore the equations for the second point are

$$\begin{cases} x = r\cos(\pi - 4\pi t), \\ y = r\sin(\pi - 4\pi t). \end{cases}$$

20. (a) Assume a unit circle, time in seconds, and angular velocity in radians per second. The 10 o'clock position, T, has an angular displacement of $\frac{5\pi}{6}$. Since point P arrives at position T in 10 seconds, its angular velocity is $\frac{5\pi}{60}$ or $\frac{\pi}{12}$. In the same way the angular velocities of Q, R, and S are $\frac{\pi}{30}$, $-\frac{\pi}{60}$, and $-\frac{2\pi}{30}$ or $-\frac{\pi}{15}$.



Therefore, as before, the equations of motion are:

$$P: \begin{cases} x = \cos \frac{\pi}{12} t , \\ y = \sin \frac{\pi}{12} t . \end{cases}$$

$$Q: \begin{cases} x = \cos(\frac{\pi}{2} + \frac{\pi}{30} t) , \\ y = \sin(\frac{\pi}{2} + \frac{\pi}{30} t) . \end{cases}$$

$$R: \begin{cases} x = \cos(\pi - \frac{\pi}{60} t) , \\ y = \sin(\pi - \frac{\pi}{60} t) . \end{cases}$$

$$S: \begin{cases} x = \cos(\frac{3\pi}{2} - \frac{\pi}{15} t) , \\ y = \sin(\frac{3\pi}{2} - \frac{\pi}{15} t) . \end{cases}$$

(b) By the methods of the solution of the previous exercise we find that the meetings of the following pairs take place at the indicated times (where a , b , c , d , are positive integers):

Q and R , when $t_1 = 10 + 40a$;

Q and S , when $t_2 = 10 + 20b$;

P and R , when $t_3 = 10 + 20c$;

P and S , when $t_4 = 10 + \frac{40}{3}d$.

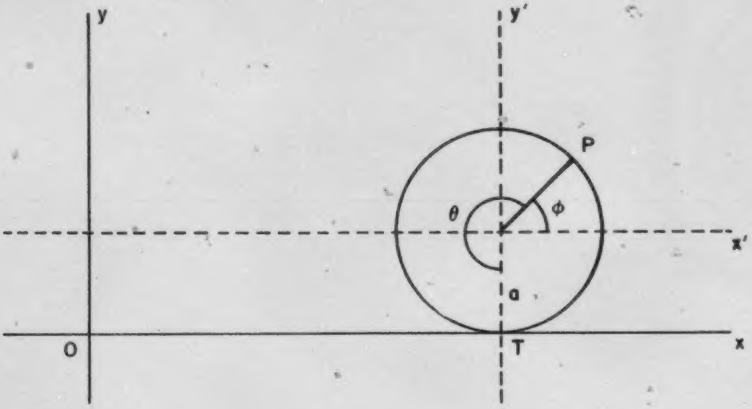
We verify that when a , b , c , d are all zero, the values of t_1 , t_2 , t_3 , t_4 , are all equal to 10 , as required by the statement of the problem. If there is to be a simultaneous meeting at another time, there must be values of a , b , c , d other than zero for which these times are equal. Clearly, if we take $d = 3$ or any multiple of 3, we can find such values. When $d = 3$, then $t_4 = 10 + 40 = 50$. Successive multiples of 3 as values of d give values of t_4 : 10 , 50 , 90 , 130 , ... , and these are clearly possible values of t_1 , t_2 , and t_3 , also. That is, the simultaneous meetings take place every 40 seconds after the first such meeting. The angular positions of these meetings are found to be $\frac{5\pi}{6}$, $\frac{25\pi}{6}$, $\frac{15\pi}{2}$, $\frac{130\pi}{12}$, ...

Questions of meeting or overtaking on circular paths are related to important problems in space exploration. Consider the complications that arise: the paths in space are not circular but essentially elliptical; the paths are not along the same ellipse, and the different ellipses are not usually in the same plane, so that we must not consider the meeting points (they would be catastrophic), but the points of nearest approach; the velocities along these paths are not uniform but variable in very complicated ways. The solutions to the exercises in our text are essential first steps in arriving at the level of ability needed to solve the difficult problems of astrogation that arise in space travel.

5-5. Parametric Equations of the Cycloid.

The physical applications of the cycloid are interesting indeed but their analysis is beyond the scope of this book. Students who are interested in photography can make photographs of a cycloid by taking a time exposure of a flashlight attached to an automobile wheel as it rolls along the road.

We give another derivation of the equations of the cycloid which uses the idea of a transformation of coordinates. You may wish to leave this derivation until you have reached the more complete treatment of transformation in Chapter 10.



Since $d(O,T)$ = length of $\widehat{PT} = a\theta$, the coordinates of the center of the circle are $(a\theta, a)$. We take this point as origin of an x'-, y'-coordinate system, hence $P = (x,y)$ becomes

$$P = (x', y'),$$

where
$$\begin{cases} x = x' + a\theta, \\ y = y' + a. \end{cases}$$

But in this new coordinate system

$$\begin{cases} x' = a \cos \phi, \\ y' = a \sin \phi. \end{cases}$$

Since $\phi = \frac{3\pi}{2} - \theta$ we have

$$\cos \phi = -\sin \theta \quad \text{and} \quad \sin \phi = -\cos \theta,$$

therefore
$$\begin{cases} x' = -a \sin \theta, \\ y' = -a \cos \theta, \end{cases}$$

Therefore, finally

$$\begin{cases} x = -a \sin \theta + a\theta, \\ y = -a \cos \theta + a; \end{cases} \quad \text{or} \quad \begin{cases} x = a(\theta - \sin \theta), \\ y = a(1 - \cos \theta). \end{cases}$$

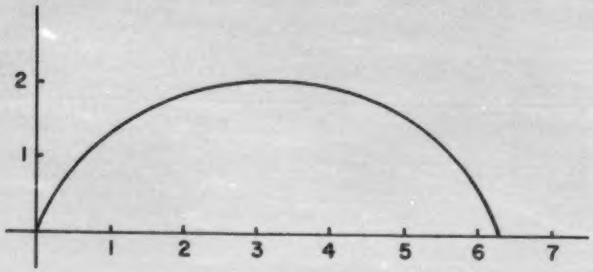
Exercises 5-5

1. $$\begin{cases} x = \theta - \sin \theta, \\ y = 1 - \cos \theta. \end{cases}$$

The intervals suggested indicate degree measure, but it would be an error to use these measures in the equations above, since the equations were derived on the basis of radian measure for θ. We may revise the formulas to suit degree measure, or convert the intervals to radian measure. The latter procedure is the easier and the one we follow.

θ degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
θ radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x	0	0	.2	.6	1.2	2.1	3.1	4.2	5.1	5.7	6.1	6.3	6.3
y	0	.1	.5	1.0	1.5	1.9	2.0	1.9	1.5	1.0	.5	.1	0

The values of x and y are computed to the nearest tenth, and the graph is sketched below.



2. The height of the rectangle is the diameter of the generating circle whose radius is therefore equal to 3 . The base of the rectangle is as long as the circumference of that circle and is therefore 6π . The equations of the cycloid are

$$\begin{cases} x = 3(\phi - \sin \phi) , \\ y = 3(1 - \cos \phi) . \end{cases}$$

3. We have $a = 3$ inches, and equations for the graph,

$$\begin{cases} x = 3(\phi - \sin \phi) , \\ y = 3(1 - \cos \phi) . \end{cases}$$

The angular velocity is given as 4 rps which means that $\omega = 8\pi$ radians per second. Since $\theta = \omega t$ the equations above become

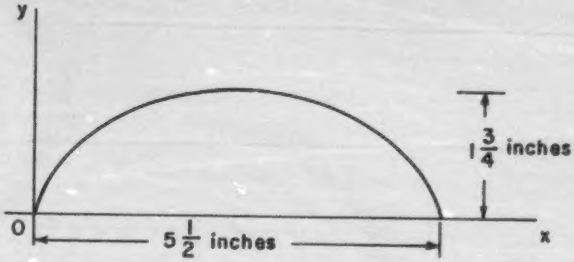
$$\begin{cases} x = 3(8\pi t - \sin 8\pi t) , \\ y = 3(1 - \cos 8\pi t) . \end{cases}$$

t	.1	.2	.3	.4	.5
x	5.77	17.93	19.75	28.38	37.68
y	5.42	2.08	2.08	5.42	0

To compute these values we had to find functions of angles whose radian measures exceeded 1.60 , which is as far as our Table II goes. We must use the procedure explained in the solution to Exercise 5-3, Number 11. Thus $\sin .8\pi = \sin 2.51 = \sin(\pi - 2.51) = \sin .63 = .589$, and so on.

P will reach its first high point at the end of the first half turn which will occur at the end of the first $\frac{1}{8}$ second. When $t = .125$, $P = (9.4 , 6)$.

4. (a) All cycloids have the same shape, therefore an accurate scale drawing requires any carefully drawn cycloid and a properly chosen scale. The width of one arch is $2\pi a$, and the height is $2a$, where a is the radius of the generating circle. In this case the base line represents 66 inches, or $2\pi a$. Therefore $a = 10\frac{1}{2}$ inches. We suggest a scale of 1:12 which means that the drawing should be $5\frac{1}{2}$ inches across and $1\frac{3}{4}$ inches high.



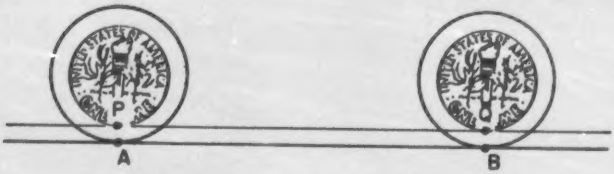
(b) We have

$$\begin{cases} x = a(\phi - \sin\phi), \\ y = a(1 - \cos\phi); \end{cases} \qquad a = 10\tfrac{1}{2}.$$

We must correct the linear rate of 30 mph into an angular rate of rotation for a wheel with 66 inch circumference. A rate of 30 mph $= \frac{30 \cdot 5280 \cdot 12}{60}$ inches per minute $= \frac{6 \cdot 5280}{66}$ rotations per minute $= \frac{5280}{11} 2\pi$ radians per minute. Therefore $\omega = \frac{10560}{11}\pi$ and $\theta = \frac{10560}{11}\pi t$. Finally we have the equations of motion with values for x and y in inches, and t in minutes:

$$\begin{cases} x = \frac{21}{2}\left(\frac{10560}{11}\pi t - \sin\frac{10560}{11}\pi t\right), \\ y = \frac{21}{2}\left(1 - \cos\frac{10560}{11}\pi t\right). \end{cases}$$

You may wish to present the following "paradox" and solicit explanations from the class:



Suppose a nickel and a dime are firmly attached concentrically, and the nickel is rolled one full turn without slipping along the line $\overleftrightarrow{AB}$. Then $d(A,B)$ is the circumference of the nickel and since $d(A,B) = d(P,Q)$ the circumferences are equal. Aren't they?

Answer. (Don't tell the class too soon.) Of course the circumferences are not equal. If the nickel doesn't slip along $\overleftrightarrow{AB}$ then the dime must slip along $\overleftrightarrow{PQ}$.

Challenge Exercises for Sections 5-3, 5-4, 5-5

1. From Figure 5-13, since $d(O,G) = \text{length of } \widehat{FG} = a\phi$, the coordinates of C are $(a\phi, a)$. If $P = (x,y)$ is a point of the locus, then

$$\begin{cases} x = a\phi - b \sin \phi , \\ y = a - b \cos \phi . \end{cases}$$

In Figure 5-14 the point Q less coordinates $(0,k)$. To find k we first find ϕ from $0 = 4\phi - 6 \sin \phi$. We can do this only approximately, from the tables and the fact that $\sin \phi = \frac{2}{3}\phi$. From Table II we have $\sin 1.50 = 0.997$ and $\sin 1.48 = 0.996$. A reasonable estimate gives $\phi \approx 1.50$, within the limits of accuracy of this table. Therefore $k \approx 4 - 6 \cos 1.50$ or $4 - 6(0.071)$. $\therefore$ $Q = (0, 3.57)$.

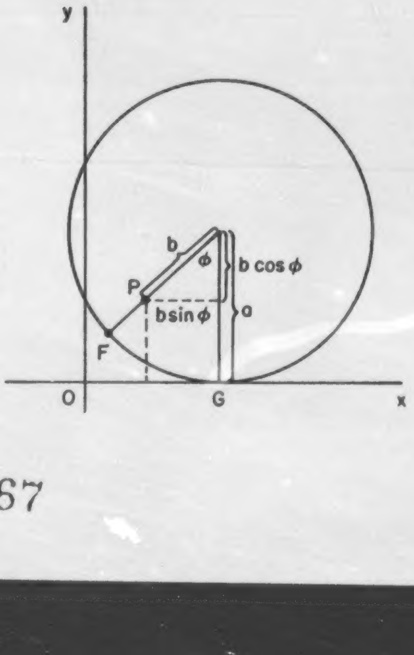
As b gets larger in comparison with a the lower loops get relatively larger, and the graph looks as if it were being compressed horizontally. The lower loops will intesect and overlap and the graph will look more and more like a plane projection of a tight helical spring, or like an elaborate doodle.

2. This drawing should make clear the relations:

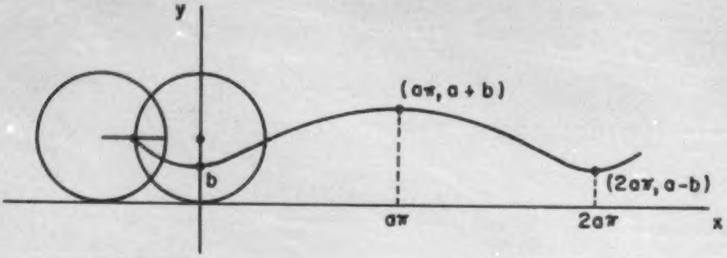
$$\begin{cases} x = d(O,G) - b \sin \phi , \\ y = a - b \cos \phi . \end{cases}$$

The equations for this curtate cycloid are exactly the same as those for the prolate cycloid.

$$\begin{cases} x = a\phi - b \sin \phi , \\ y = a - b \cos \phi . \end{cases}$$



The distinguishing feature for their graphs is in the relative sizes of a and b , as indicated in the text.



3. (Refer to Figure 5-15 in the text.) Since length of $\widehat{AB}$ = length of $\widehat{BP}$, We have $a\phi = b\theta$. Also, $C = \Big((a + b) \cos \theta , (a + b) \sin \theta\Big)$. If $P = (x,y)$ is a point of the locus then

$$\begin{cases} x = d(O,E) - d(P,D) = (a + b) \cos \theta - a \sin \psi , \\ y = d(C,E) - d(C,D) = (a + b) \sin \theta - a \cos \psi . \end{cases}$$

Since $\theta + \phi + \psi = \frac{\pi}{2}$ we have $\sin \psi = \cos(\theta + \phi)$, and $\cos \psi = \sin(\theta + \phi)$, thus we may eliminate ψ from the equations above and write

$$\begin{cases} x = (a + b) \cos \theta - a \cos(\theta + \phi) , \\ y = (a + b) \sin \theta - a \sin(\theta + \phi) . \end{cases}$$

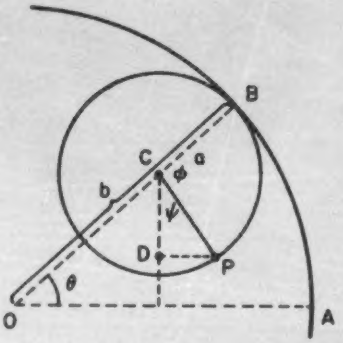
Finally, since $\phi = \frac{b}{a}\theta$ we may eliminate ϕ from the equations above and get

$$\begin{cases} x = (a + b) \cos \theta - a \cos(\theta + \frac{b}{a}\theta) , \\ y = (a + b) \sin \theta - a \sin(\theta + \frac{b}{a}\theta) . \end{cases}$$

These are usually written

$$\begin{cases} x = (a + b) \cos \theta - a \cos(\frac{a + b}{a} \theta) , \\ y = (a + b) \sin \theta - a \sin(\frac{a + b}{a} \theta) . \end{cases}$$

4. The analysis here is closely related to that of the previous solution. We furnish a diagram and essential steps only.



$$a\phi = b\theta \ .$$

$$\phi + \psi - \theta = \frac{\pi}{2}$$

$$d(P,D) = a \sin \psi \ ,$$

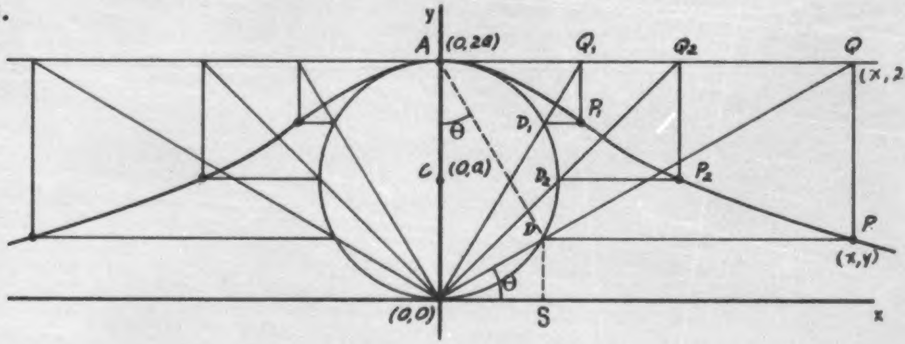
$$d(C,D) = a \cos \psi \ .$$

$$P = (x,y)$$

$$\begin{cases} x = (b - a) \cos \theta + a \sin \psi \ , \\ y = (b - a) \sin \theta - a \cos \psi \ . \end{cases}$$

$$\begin{cases} x = (b - a) \cos \theta + a \cos\left(\frac{b - a}{a}\theta\right) , \\ y = (b - a) \sin \theta - a \sin\left(\frac{b - a}{a}\theta\right) . \end{cases}$$

5.



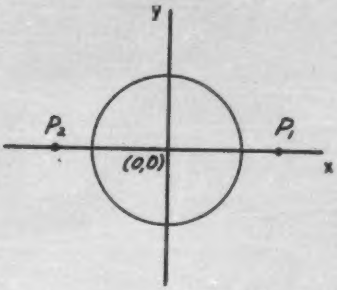
Symmetric in y-axis $0 \le y \le 2a$, x covers all reals asymptotic to x-axis, tangent to $y = 2a$. To get the analytic representation, connect points D , A . Draw $\overline{DS}$ $\perp$ to the x-axis. Then in ($\angle$SOD ($\angle$SOD) = θ = m($\angle$DAO) ; $y = d(D,S) = d(O,D) \sin \theta$; $d(O,D) = 2a \sin \theta$. Therefore, $y = 2a \sin^2\theta$. Also $x = 2a \cot \theta$. These are parametric equations for the graph,

$$\begin{cases} x = 2a \cot \theta \ , \\ y = 2a \sin^2\theta \ . \end{cases}$$

To eliminate the parameter we may square both members of the first equation and then combine with the second to obtain eventually,

$$x^2y = 4a^2(2a - y) \ , \text{ or } \ y = \frac{8a^3}{x^2 + 4a^2} \ .$$

6. Choose coordinate system so that $P_1 = (b,0)$, $P_2 = (-b,0)$. Then we get the condition $x^2 + y^2 = a^2 - b^2$. If $|a| < |b|$, there are no points in locus. If $|a| = |b|$, the locus is the point $(0,0)$. If $|a| > |b|$, the locus is a circle with origin at $(0,0)$ and radius $\sqrt{a^2 - b^2}$.



7. Square $(a,a)(-a,a)(a,-a)(-a,-a)$, constant $4k^2$, $x^2 + y^2 = k^2 - 2a^2$. If $k^2 < 2a^2$, locus is empty set. If $k^2 = 2a^2$, locus is point at $(0,0)$. If $k^2 > 2a^2$, locus is a circle with center $(0,0)$ and radius $\sqrt{k^2 - 2a^2}$.

8. Same square: side $x = a$, $x = -a$, $y = a$, $y = -a$, constant $4k^2$, $x^2 + y^2 = 2k^2 - 2a^2$. If $k^2 < a^2$, locus is empty set. If $k^2 = a^2$, locus is $(0,0)$. If $k^2 > a^2$, locus is circle with center $(0,0)$ and radius $\sqrt{2k^2 - 2a^2}$.

9. $(2c)x + (a + b)y = c(a + b)$ (The sides of the triangle may be extended to allow values of y and x outside of the triangle.)

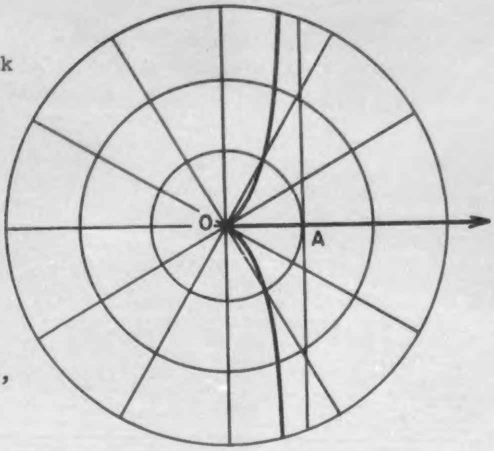
10. $y^2 + (x - \frac{9}{2})^2 = (\frac{9}{2})^2$ Q does lie on the locus.

11. (Refer to Figure 5-17 in the text.)

$$d(P,S) = d(O,R) = 2a \cos \theta , \quad \text{from right } \triangle OAR .$$
$$d(O,S) = 2a \sec \theta . \quad \text{Therefore}$$
$$r = d(O,P) = d(O,S) - d(P,S) = 2a(\sec \theta - \cos \theta) .$$

This is a polar equation for the graph. An equivalent form for this equation is $r = 2a \sin \theta \tan \theta$. To change to rectangular coordinates it is convenient to multiply both members by r and obtain $r^2 = 2a(r \sin \theta)(\tan \theta)$, which yields $x^2 + y^2 = 2a(y)(\frac{y}{x})$, which can be written, $x(x^2 + y^2) = 2ay^2$, or $y^2 = \frac{x^3}{2a - x}$.

The procedure of multiplying both members of the equation by r is convenient, but we must check that the graphs of $r = 2a \sin \theta \tan \theta$ and $r^2 = 2ar \sin \theta \tan \theta$ are the same. The only points that might be on the graph of the latter but not on that of the former are points for which $r = 0$, but the pole, which is the only such point, is already on that graph. The equations therefore do have the same graphs. The idea will escape the students unless they think about such simple examples as $x = y$ and $x^2 = xy$, whose graphs are different. The situation for polar coordinates can be stated as follows. Suppose the pole lies on the graph of the equation $f(r,\theta) = 0$. Then the graphs of that equation and the equation $rf(r,\theta) = 0$ are identical. The same thing can occur when we are dealing with rectangular coordinates. For example, the equations $x^2 = xy$ and $x^3 = x^2y$ have the same graph. The explanation is essentially the same as it was for polar coordinates. All the points which would otherwise have been added to the graph when we multiplied both members of its equation by x, were already points of the graph of $x^2 = xy$.



12. (Refer to Figure 5-18 of the text.)

A polar equation for the locus of R is $r = \frac{a}{\cos \theta}$. Therefore equations for the loci of P and P' are

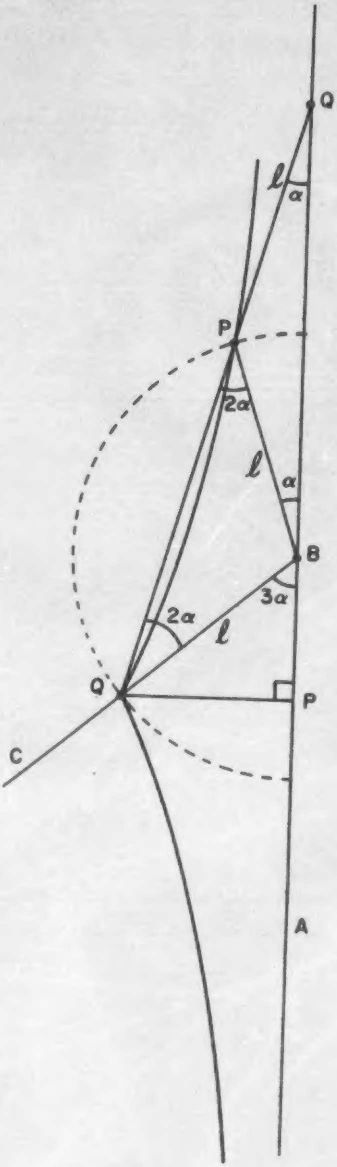
$$r = \frac{a}{\cos \theta} \pm \ell .$$

The trisection of an angle is one of the great classical problems in mathematics under the usual conditions, allowing only compasses and unmarked straightedge, the problem is provably insoluble. (See e.g., What is Mathematics, Courant and Robbins.) However, by the use of special curves which cannot be drawn solely with compasses and unmarked straightedge the problem can be solved. Any such curve used for this purpose is called a trisectrix. To show the use of the conchoid as a trisectrix we proceed as follows:

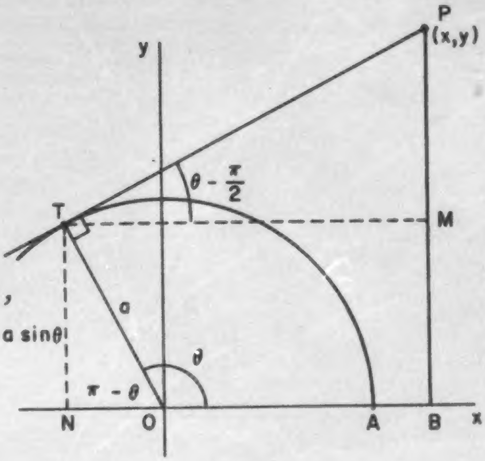
We are given any $\angle ABC$. From O, any point in $\overrightarrow{BC}$, draw $\overline{OR} \perp \overleftrightarrow{AB}$. Construct the left branch of the conchoid as in the text, using d(O,B) as length ℓ. (This is the step which is barred under the classic restriction.) Now construct a circle with B as center, and ℓ as radius, to cut the conchoid at P. Draw $\overrightarrow{OP}$ to cut $\overleftrightarrow{AB}$ at Q. We assert that $m(\angle OQA) = \frac{1}{3} m(\angle OBA)$.

Proof: Draw $\overline{PB}$. Then, from isosceles triangles PQB and PBO we can verify the relations indicated in the diagram.

Note that if ℓ is greater than the distance from the point to the line, then the left branch of the conchoid has a loop, as in the text. If ℓ equals the distance from the point to the line then the left branch has a cusp as in the illustration here. If ℓ is less than the distance from the point to the line, the left branch will have an indentation toward the fixed point.



13. (Refer to Figure 5-19.)



Through T draw lines parallel to the axes as indicated.

$$d(P,T) = a\theta \ ;$$

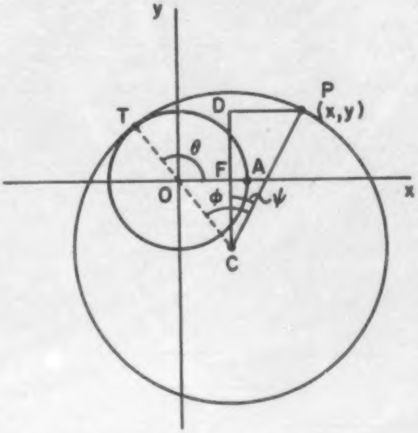
$$\begin{cases} x = d(T,M) - d(O,N) \ , \\ y = d(P,M) + d(T,N) \ . \end{cases}$$

$$\begin{cases} x = a\theta \cos(\theta - \frac{\pi}{2}) + a \cos(\pi - \theta) , \\ y = a\theta \sin(\theta - \frac{\pi}{2}) + a \sin \theta \ . \end{cases}$$

Therefore

$$\begin{cases} x = a \cos \theta + a\theta \sin \theta \ , \\ y = a \sin \theta - a\theta \cos \theta \ . \end{cases}$$

14. Students sometimes refer to this problem as the "hula-hoop" problem. Figure 5-20 in the text contains lines which are not pertinent to this solution. Please ignore them and refer to the figure at the right:



$d(C,T) = d(C,P) = b$; $d(O,T) = a$;
$d(O,C) = b - a$.

$\widehat{AT} = \widehat{PT}$, $a\theta = b$, $\phi = \frac{a}{b}\theta$.
$P = (x,y)$.

$$\begin{cases} x = d(O,F) + d(D,P) = (b - a) \sin (\phi - \psi) + b \sin \psi \ , \\ y = d(D,C) - d(F,C) = b \cos \psi - (b - a) \cos (\phi - \psi) \ . \end{cases}$$

Since $\theta = \phi - \psi + \frac{\pi}{2}$, we may eliminate ψ :

$$\sin(\phi - \psi) = \sin(\theta - \frac{\pi}{2}) = -\cos \theta \ ,$$

$$\cos(\phi - \psi) = \cos(\theta - \frac{\pi}{2}) = \sin \theta \ ;$$

$$\sin \psi = \sin(\phi - \theta + \frac{\pi}{2}) = \cos(\phi - \theta) = \cos(\theta - \phi) \ ,$$

$$\cos \psi = \cos(\phi - \theta + \frac{\pi}{2}) = -\sin(\phi - \theta) = \sin(\theta - \phi) \ .$$

Therefore, $\begin{cases} x = -(b - a) \cos \theta + b \cos(\theta - \phi) , \\ y = b \sin(\theta - \phi) - (b - a) \sin \theta . \end{cases}$

Finally, since $\phi = \frac{a}{b}$,

$$\begin{cases} x = -(b - a) \cos \theta + b \cos \left(\frac{b - a}{b} \theta\right) , \\ y = -(b - a) \sin \theta + b \sin \left(\frac{b - a}{b} \theta\right) . \end{cases}$$

5-6. Parametric Equations of a Straight Line.

The material in this section uses methods developed in this chapter to extend and apply the content introduced in Chapter 2. We recommend here and throughout the book that students be required to refer backwards and forwards. To prepare for this section students should be given, in the preceding few days, some home-work exercises from the latter half of Chapter 2, and that you continue giving some home-work exercises from that chapter as you go on through this section. A systematic overlapping of such assignments is a feature of what is called "spiral" assignments, which we recommend.

The geometric version of the assumption that $x_1 = x_0$ is that the two points are equidistant from the y-axis, the geometric version of the conclusion (that the equations are $x = x_0$, $y = y_0 + mt$) , is that the line through these points is parallel to the y-axis. In the second case the assumption is equivalent to saying that the points are equidistant from the x-axis, and the conclusion is equivalent to saying that the line through them is parallel to the x-axis.

It makes no difference what letter is used for the parameter in parametric equations for a line. Thus we could have represented the lines L_1 and L_2 of Example 2 as follows:

$$L_1 : \begin{aligned} x &= 4 - 2t \\ y &= 2 - 6t \end{aligned} \qquad\qquad L_2 : \begin{aligned} x &= -3 - t \\ y &= -1 + 3t \end{aligned}$$

If a student asks whether the two t's are equal, it must be made clear that the question is meaningless. They are both variables and can take any real value. Suppose we had used the representations above and had then tried to find the intersection of the lines by solving the simultaneous equations

$$4 - 2t = -3 - t$$
$$2 - 6t = -1 + 3t.$$

The question we would really have been trying to answer is whether there are any values of t which give the same point on both lines, and this is not the question we started with. This point comes up again in Example 3.

Exercises 5-6

1. (a) $\begin{cases} x = 5 - 3t \\ y = -1 + 4t \end{cases}$ $\quad\begin{cases} x = 2 + 3t \\ y = 3 - 4t \end{cases}$

   (b) $\begin{cases} x = 0 + 4t \\ y = 0 + 1t \end{cases}$ $\quad\begin{cases} x = 4 - 4t \\ y = 1 - 1t \end{cases}$

   (c) $\begin{cases} x = 2 + 0t \\ y = -3 + 6t \end{cases}$ $\quad\begin{cases} x = 2 - 0t \\ y = 3 - 6t \end{cases}$

   (d) $\begin{cases} x = -1 - 5t \\ y = 4 + 0t \end{cases}$ $\quad\begin{cases} x = -6 + 5t \\ y = 4 + 0t \end{cases}$

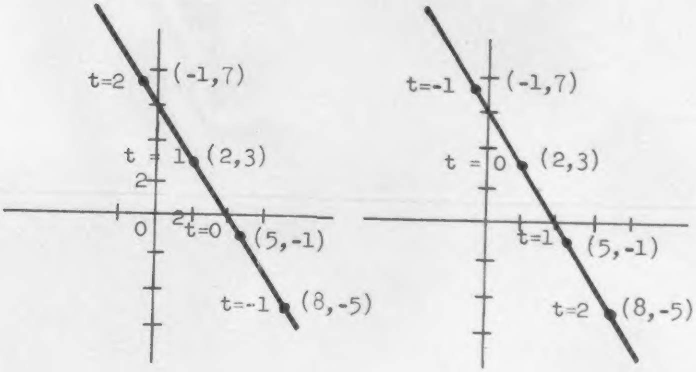
   (e) $\begin{cases} x = 1 + 1 \cdot t \\ y = 1 + 1 \cdot t \end{cases}$ $\quad\begin{cases} x = 2 - 1 \cdot t \\ y = 2 - 1 \cdot t \end{cases}$

   (f) $\begin{cases} x = -1 + 2t \\ y = -1 + 2t \end{cases}$ $\quad\begin{cases} x = 1 - 2t \\ y = 1 - 2t \end{cases}$

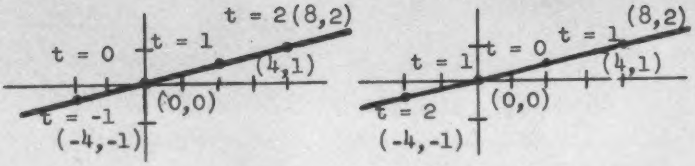
   (g) $\begin{cases} x = 1 - 1 \cdot t \\ y = 0 + 1 \cdot t \end{cases}$ $\quad\begin{cases} x = 0 + 1 \cdot t \\ y = 1 - 1 \cdot t \end{cases}$

   (h) $\begin{cases} x = 2 - 4t \\ y = -2 + 4t \end{cases}$ $\quad\begin{cases} x = -2 + 4t \\ y = 2 - 4t \end{cases}$

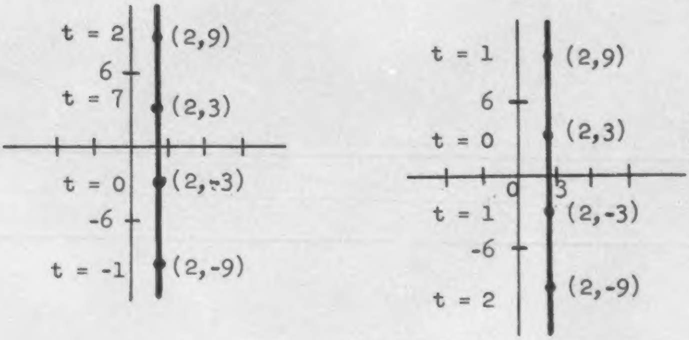
2. (a)



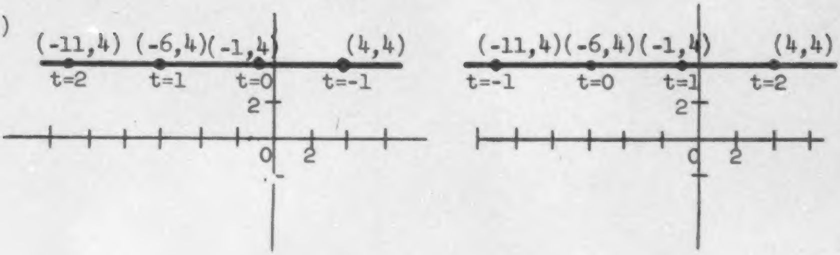
(b)


t = 2 (8,2)
t = 0
t = 1
(4,1)
(0,0)
t = -1
(-4,-1)
(8,2)
t = 1
t = 0
t = 1
(4,1)
(0,0)
t = 2
(-4,-1)

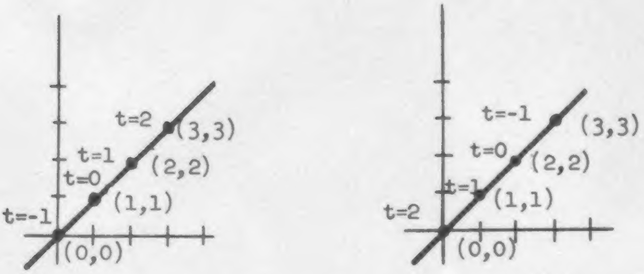
(c)


t = 2
(2,9)
6
t = 7
(2,3)
t = 0
(2,-3)
-6
t = -1
(2,-9)
t = 1
(2,9)
6
t = 0
(2,3)
0
3
t = 1
(2,-3)
-6
t = 2
(2,-9)

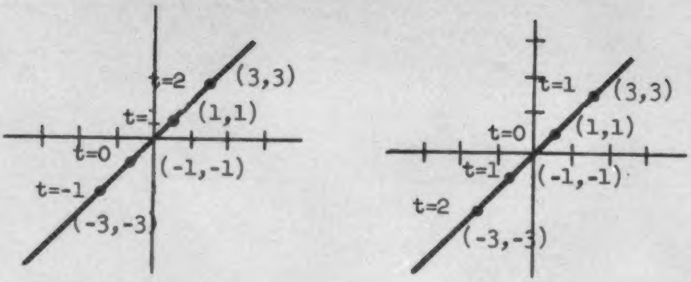
(d)


(-11,4)
(-6,4)
(-1,4)
(4,4)
t=2
t=1
t=0
t=-1
2
0
2
(-11,4)
(-6,4)
(-1,4)
(4,4)
t=-1
t=0
t=1
t=2
2
0
2

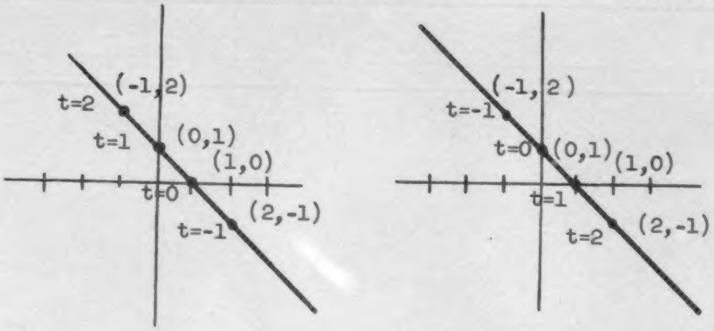
(e)


t=2
(3,3)
t=1
(2,2)
t=0
(1,1)
t=-1
(0,0)
t=-1
(3,3)
t=0
(2,2)
t=1
(1,1)
t=2
(0,0)

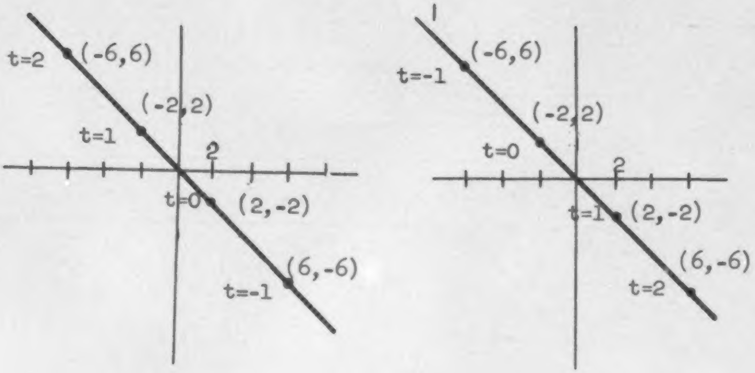
(f)



(g)



(h)



3. (a) $(-14,21)$

(b) The lines are parallel; their pairs of direction numbers are equivalent: $(6,-4) = (-2(-3), -2(2))$

(c) The lines are coincident; their pairs of direction numbers are equivalent and they have at least one point $(-3,2)$ in common.

4. Using points $(1,1)(4,3)$ on the line $L : 2x - 3y + 1 = 0$.

$$x = 1 + 3t$$
$$y = 1 + 2t \ .$$

5. $x_1 - x_2 = \ell(t_1 - t_2), \quad y_1 - y_2 = m(t_1 - t_2)$

$$d(P_1, P_2) = \sqrt{\ell^2(t_1 - t_2)^2 + m^2(t_1 - t_2)^2}$$
$$= \sqrt{(t_2 - t_1)^2} \cdot \sqrt{\ell^2 + m^2}$$
$$= |t_2 - t_1| \cdot \sqrt{\ell^2 + m^2}$$

6. $\begin{cases} x = 16 + t(-24) \\ y = 2 + t(10) \end{cases}$

7. (a) Substituting $x = \lambda t, \ y = \mu t$ into $ax^2 + by^2 = a^2b^2$ gives

$$a\lambda^2 t \ + b\mu^2 t^2 = a^2b^2$$
$$t^2(a\lambda^2 + b\mu^2) = a^2b^2$$

If $\quad a\lambda^2 + b\mu^2 \neq 0$

$$t^2 = \frac{a^2b^2}{a\lambda^2 + b\mu^2} \ ;$$

if $\quad a\lambda^2 + b\mu^2 > 0$

$$t = \pm \frac{|ab|}{\sqrt{a\lambda^2 + b\mu^2}} \ ;$$

hence line intersects figure at points equidistant from 0 under conditions mentioned.

(b) Putting $x = \lambda t$, $y = \mu t$ into $y = ax^3$, we get $t = a\lambda^3 t^3$. If $a > 0$ for $\mu \neq 0$, $\lambda \neq 0$ and considering only $t \neq 0$ we get $t^2 = \frac{\mu}{a\lambda^3}$.

If $\mu \cdot \lambda > 0$, $t = \pm \frac{1}{\lambda}\sqrt{\frac{\mu}{a\lambda}}$ and intersections are symmetric.

If $\mu \cdot \lambda < 0$, there are no intersections for $t \neq 0$.

Thus the origin is the center.

(2) $a < 0$, for $\mu \neq 0$, $\lambda \neq 0$ and considering $t \neq 0$ we get $t^2 = \frac{\mu}{a\lambda^3}$.

If $\mu \cdot \lambda > 0$, there are no intersections for $t \neq 0$.
If $\mu \cdot \lambda < 0$, then there are intersections for

$$t = \pm \frac{1}{\lambda} \sqrt{\frac{\mu}{a\lambda}} .$$

Again the origin is the center.

(c) Putting $x = \lambda t$, $y = \mu t$ into $y = \frac{x^3}{x^2 - 1}$

we get $\mu t = \frac{\mu^3 t^3}{\lambda^2 t^2 - 1}$ which is not defined for $\lambda t = 1$

$$\mu\lambda^2 t^3 - \mu t = \mu^3 t^3 .$$

If $\mu \neq 0$ $\qquad \lambda^2 t^3 - t = \mu^2 t^3$;

if $t \neq 0$ $\qquad \lambda^2 t^2 - 1 = \mu^2 t^2$;

$$\therefore \; t^2(\lambda^2 - \mu^2) = 1 .$$

If $\mu^2 \neq \lambda^2$ $\qquad t^2 = \frac{1}{\lambda^2 - \mu^2}$.

If $\lambda^2 > \mu^2$, then the line intersects the curve for

$$t = \pm\sqrt{\frac{1}{\lambda^2 - \mu^2}}$$, that is, symmetrically.

There is no value of t if $\lambda^2 \leq \mu^2$. Thus the curve has the origin as its center.

8. We suppose that a bounded set S has two centers, and show that we get a contradiction. We call these centers O and I and establish a coordinate system with origin at O , with x-axis along $\overleftrightarrow{OI}$, and I as the point $(1,0)$. If O and I are centers then O has a symmetric image, O_1 in I , and $O_1 = (2,0)$. O_1 has a symmetric image O_2 in O_1 and $O_2 = (-2,0)$. O_2 has a symmetric image O_3 in I , and $O_3 = (3,0)$, and so on. The points O_1 , O_3 , O_5 , ... , are all members of S and their coordinates, $(2,0)$, $(3,0)$, $(4,0)$, ... , indicate that they are farther and farther from the origin. Clearly they cannot all be enclosed by an finite rectangle, which means that S cannot be bounded.

The statement is not true for unbounded sets; for example any point of a line is a center of the set of points of that line.

9. We express the line in parametric form using direction cosines:

$$\begin{cases} x = 5 + 0.8t \ , \\ y = 8 + 0.6t \ . \end{cases}$$

When $t = 1$, $(x,y) = (5.8, 8.6)$;
when $t = -1$, $(x,y) = (4.2, 7.4)$.

10. $\begin{cases} x = 0 + 3t \ , \\ y = 9 + 4t \ . \end{cases}$

It is simplest here to use $d(A,B)$ units along the line. When $t = 5$, $(x,y) = (15,29)$; when $t = -5$, $(x,y) = (-15,-11)$.

Review Exercises

In the answers to these exercises we supply, in most cases, the simplest and most directly achieved answer. It is always to be understood that a given graph has infinitely many analytic representations. Some of these may be trivially related as: $y = 5$ and $2y = 10$; some non-trivally as: $x + 2y - 11 = 0$ and

$$\begin{cases} x = 5 + 4t \ , \\ y = 3 - 2t \ . \end{cases}$$

The teacher is particularly urged in this chapter to consider carefully any pupil's answer which may differ from the one presented here. It may be correct, but written in unfamiliar form, and the student may, with benefit, carry the burden of showing the equivalence of the two.

When we are asked for an analytic description of a set, for example, 2(a) below, we will usually write our answer in the form in which it appears in the literature:

$$x - 4y + 7 = 0 \ ,$$

instead of the longer form:

$$\{(x,y) : x - 4y + 7 = 0\} \ .$$

1. (a) The lines: $y = x$ and $y = -x$; or $y^2 = x^2$.
   (b) The line: $x = 8$.
   (c) The line: $y = 4$.
   (d) The line: $3x - 4y - 8 = 0$.
   (e) The circle: $(x - 5)^2 + (y - 8)^2 = 9$, which can also be written: $x^2 + y^2 - 10x - 16y + 80 = 0$.
   (f) The lines: $x = 2$ and $x = 8$.

(g) The lines: $y = 1$ and $y = -5$.

(h) The lines: $3x - 4y + 22 = 0$ and $3x - 4y - 8 = 0$.

(i) The lines: $x = k + h$ and $x = k - h$.

(j) The lines: $y = q + p$, $y = q - p$.

(k) If $ax + by + c = 0$ represents a line, then $a^2 + b^2 \neq 0$ and the distance from $P = (x_0,y_0)$ to this lines is given by $d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$. This equation is equivalent to $ax_0 + by_0 + c = \pm d\sqrt{a^2 + b^2}$, therefore the locus of all such points $P = (x,y)$ is the pair of lines represented by $ax + by + c + d\sqrt{a^2 + b^2} = 0$, and $ax + by + c - d\sqrt{a^2 + b^2} = 0$.

(l) The distance from $P = (x,y)$ to $A = (5,0)$ is $\sqrt{(x - 5)^2 + y^2}$, and to $B = (11,0)$ is $\sqrt{(x - 11)^2 + y^2}$. The condition is equivalent to; $\sqrt{(x - 5)^2 + y^2} = 2\sqrt{(x - 11)^2 + y^2}$. This equation is an answer to the exercise, but it can be written more simply as $x^2 + y^2 - 26x + 143 = 0$, or as $(x - 13)^2 + y^2 = 4^2$. This last equation yields the additional information that the graph is a circle with center at $(13,0)$ and with radius 4 .

(m) The condition yields directly: $y = \sqrt{(x - 5)^2 + (y - 8)^2}$ or more simply $x^2 - 10x - 16y + 89 = 0$. This can also be written $(x - 5)^2 = 16(y - 4)$, which can be interpreted to be an equation of a parabola with vertex at $(5,4)$, axis along the y-axis, and open upward.

(n) As above, we get the parabola: $y^2 - 8x + 24 = 0$.

(o) The distance from $P = (x,y)$ to $D = (5,3)$ is $\sqrt{(x - 5)^2 + (y - 3)^2}$. The distance from $P = (x,y)$ to the line $3x - 4y + 7 = 0$ is $\frac{|3x - 4y + 7|}{\sqrt{3^2 + 4^2}}$. An answer to this exercise is given by the statement of equality for these two distances, $\sqrt{(x - 5)^2 + (y - 3)^2} = \frac{|3x - 4y + 7|}{\sqrt{3^2 + 4^2}}$. This can be written somewhat more simply as $16x^2 + 24 + y + 9y^2 - 292x - 94y + 801 = 0$. We state that the graph is a parabola with an oblique axis perpendicular to the given line, but we leave any further discussion of this equation and graph for Chapter 10.

(p) As in the previous exercise, an answer is given by:

$$\sqrt{(x - r)^2 + (y - s)^2} = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$$, which can be written also as:

$(ax + by + c)^2 = (a^2 + b^2)\ (x - r)^2 + (y - s)^2$, or, as a polynomial in x and y :

$$b^2x^2 - 2abxy + a^2y^2 - 2(ac + a^2r + b^2r)x - 2(bc + a^2s + b^2s)y$$
$$+ (a^2r^2 + a^2s^2 + b^2r^2 + b^2s^2 - c^2) = 0 .$$

We state again without proof that the graph of this equation is a parabola with its axis perpendicular to the given line.

2. In (a) - (i) we give our answers in both rectangular and parametric forms; either or both may be used.

(a) $x - 4y + 7 = 0$; or $\begin{cases} x = -3 + 8t , \\ y = 1 + 2t . \end{cases}$

(b) $x - 4y + 7 = 0 , \ x \geq -3$; or $\begin{cases} x = -3 + 8t , \\ y = 1 + 2t , \end{cases} \quad t \geq 0 .$

(c) $x - 4y + 7 = 0 , \ -3 \leq x \leq 5$; or $\begin{cases} x = -3 + 8t , \\ y = 1 + 2t , \end{cases} \quad 0 \leq t \leq 1.$

(d) $x + 2y - 11 = 0$; or $\begin{cases} x = 5 - 4t , \\ y = 3 + 2t . \end{cases}$

(e) $x + 2y - 11 = 0 , \ x \leq 5$; or $\begin{cases} x = 5 - 4t , \\ y = 3 + 2t , \end{cases} \quad t \geq 0 .$

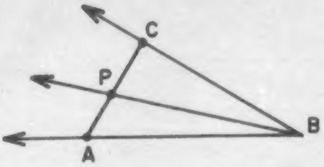
(f) $x + 2y - 11 = 0 , \ 1 \leq x \leq 5$; or $\begin{cases} x = 5 - 4t , \\ y = 3 + 2t , \end{cases} \quad 0 \leq t \leq 1.$

(g) $x - y + 4 = 0$; or $\begin{cases} x = 1 - 4t , \\ y = 5 - 4t . \end{cases}$

(h) $x - y + 4 = 0 , \ x \leq 1$; or $\begin{cases} x = 1 - 4t , \\ y = 5 - 4t , \end{cases} \quad t \geq 0 .$

(i) $x - y + 4 = 0 , \ -3 \leq x \leq 1$; or $\begin{cases} x = 1 - 4t , \\ y = 5 - 4t , \end{cases} \quad 0 \leq t \leq 1.$

(j) This, and the next four parts of this exercise are most readily done with parametric representations or vectors. The interior of $\angle ABC$ can be described as the set of points of the interior of all rays $\overrightarrow{BP}$, where P is a point of the interior of $\overline{CA}$. In that case



$P = (x,y)$; where $x = 1 - 4t$, $y = 5 - 4t$, $0 < t < 1$, from (i) above. We need another parameter to give us the interior of $\overrightarrow{BP}$. Thus direction numbers for $\overrightarrow{BP}$ are $(1 - 4t - 5 , 5 - 4t - 3)$, or $(-4 - 4t , 2 - 4t)$. Thus, for a point $Q = (x,y)$ of the interior of $\overrightarrow{BP}$ we have $x = 5 + s(-4 - 4t)$, $y = 3 + s(2 - 4t)$, $s > 0$. We present this answer more neatly:

$\{(x,y) : x = 5 - 4x - 4st , y = 3 + 2s - 4st , s > 0 , 0 < t < 1\}$.

In vector form, if P is an interior point of $\overline{CA}$ then $\overrightarrow{p} = \overrightarrow{c} + t(\overrightarrow{a} - \overrightarrow{c})$, $0 < t < 1$. If Q is an interior point of $\overrightarrow{BP}$, then $\overrightarrow{q} = \overrightarrow{b} + s(\overrightarrow{p} - \overrightarrow{b})$, $s > 0$. In terms of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, we have $\overrightarrow{q} = \overrightarrow{b} + s\left(\overrightarrow{c} + t(\overrightarrow{a} - \overrightarrow{c}) - \overrightarrow{b}\right)$, $\overrightarrow{q} = (st)\overrightarrow{a} + (1 - s)\overrightarrow{b} + (s - st)\overrightarrow{c}$, with $s > 0$, $0 < t < 1$. Note that the sum of the scalar multipliers is 1 .

We can show the equivalence of the vector and parametric forms by expressing each vector in terms of its components and then combining, retaining the parametric conditions $s > 0$, $0 < t < 1$. Thus: $\overrightarrow{q} = [x,y]$, $\overrightarrow{a} = [-3,1]$, $\overrightarrow{b} = [5,3]$, $\overrightarrow{c} = [1,5]$. Then

$[x,y] = st[-3,1] + (1 - s)[5,3] + (s - st)[1,5]$,

$[x,y] = [-3st + 5 - 5s + s - st , st + 3 - 3s + 5s - 5st]$,

$[x,y] = [5 - 4s - 4st , 3 + 2s - 4st]$.

Therefore

$$\begin{cases} x = 5 - 4s - 4st , \\ y = 3 + 2s - 4st ; \end{cases}$$

and these are the parametric equations we found before.

(k) If P is a point of the interior of $\overline{AB}$, then $P = (-3 + 8t , 1 + 2t)$, $0 < t < 1$. Proceed as in the previous solution and obtain the answer,

$\{(x,y) : x = 1 - 4s + 8st , y = 5 - 4s + 2st , s > 0 , 0 < t < 1\}$.

In vector form $\overrightarrow{p} = \overrightarrow{a} + t(\overrightarrow{b} - \overrightarrow{a})$, $0 < t < 1$, and $\overrightarrow{q}$, the vector to any point Q of the interior of $\angle BCA$ is given by

$\overrightarrow{q} = \overrightarrow{c} + s(\overrightarrow{p} - \overrightarrow{c})$, $s > 0$. This can be written in terms of $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ as was done in the previous solution:
$\overrightarrow{q} = (s - st)\overrightarrow{a} + (st)\overrightarrow{b} + (1 - s)\overrightarrow{c}$, $s > 0$, $0 < t < 1$.
Note the resemblance to the result in the previous exercise. The component forms of these vectors can be used to relate this result to the parametric equation found a few lines earlier.

(ℓ) (Refer to the two previous solutions.)
$\overrightarrow{p} = \overrightarrow{c} + t(\overrightarrow{b} - \overrightarrow{c})$, $0 < t < 1$; $\overrightarrow{q} = \overrightarrow{a} + s(\overrightarrow{p} - \overrightarrow{a})$, $s > 0$.
$\overrightarrow{q} = (1 - s)\overrightarrow{a} + (st)\overrightarrow{b} + (s - st)\overrightarrow{c}$, $s > 0$, $0 < t < 1$.
The parametric form is
$\{(x,y) : x = -3 + 4s + 4st ,\ 1 + 4s - 2st ,\ s > 0 ,\ 0 < t < 1\}$.

(m) The interior of $\triangle ABC$ is part of the interior of $\angle ABC$. If we refer to the solution of part (j) of this group we need now use only the interior points of $\overline{BP}$ where P is an interior point of $\overline{AC}$. We can effect this result by a simple change on the parameter s which we now take $0 < s < 1$. Our solution in vector form is therefore:
$\overrightarrow{q} = (st)\overrightarrow{a} + (1 - s)\overrightarrow{b} + (s - st)\overrightarrow{c}$, with $0 < s < 1$, $0 < t < 1$.
We could use the results of (k) and (ℓ) above, and obtain
$\overrightarrow{q} = (s - st)\overrightarrow{a} + (st)\overrightarrow{b} + (1 - s)\overrightarrow{c}$, $0 < s < 1$, $0 < t < 1$;
$\overrightarrow{q} = (1 - s)\overrightarrow{a} + (st)\overrightarrow{b} + (s - st)\overrightarrow{c}$, $0 < s < 1$, $0 < t < 1$.

The similarity of these expressions leads to a more symmetric formula, if we note that the scalar multipliers are non-negative and have the sum 1 . We may write a vector formula for the interior of $\triangle ABC$ thus:
$\overrightarrow{q} = \alpha\overrightarrow{a} + \beta\overrightarrow{b} + \gamma\overrightarrow{c}$, where α , β , γ are non-negative and $\alpha + \beta + \gamma = 1$.

(n) $x + 2y + 1 = 0$.

(o) $x - y - 2 = 0$.

(p) $x - 4y + 19 = 0$.

(q) $2x - y + 7 = 0$.

(r) $x + y - 8 = 0$.

(s) $4x + y - 9 = 0$.

(t) $x - 2y + 5 = 0$.

(u) $y = 3$.

(v) $x = 1$.

(w) The line $y = 1$ is parallel to the x-axis, and the line $x = -3$ is parallel to the y-axis.

(x) $4x + y - 6 = 0$.

(y) $2x - y - 2 = 0$.

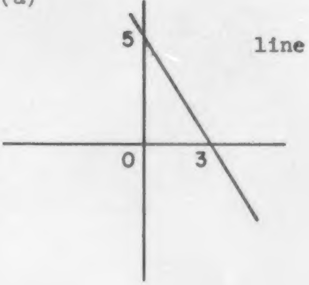
(z) If the center of the circle is at (u,v) then

$$(1-u)^2 + (5-v)^2 = (5-u)^2 + (3-v)^2 = (3+u)^2 + (1-v)^2 = r^2$$

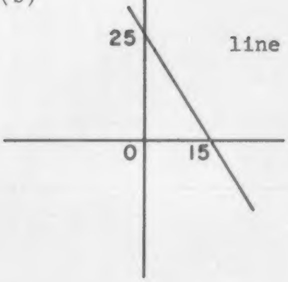
Solving these equations gives the coordinates of the center, $(\frac{4}{3}, \frac{2}{3})$, and the length of the radius, $\frac{\sqrt{170}}{3}$. Thus the circle has the equation, $(x - \frac{4}{3})^2 + (y - \frac{2}{3})^2 = \frac{170}{9}$, which may be written also as $3x^2 + 3y^2 - 8x - 4y - 50 = 0$.

3. The abbreviated sketch we supply for each part of this exercise should indicate the answers requested originally. Other brief comments are supplied as seem necessary.

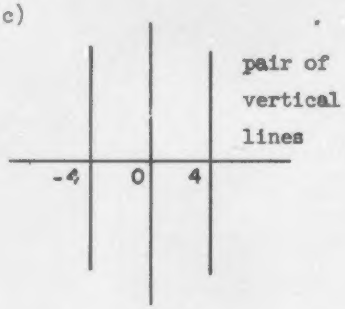
(a)



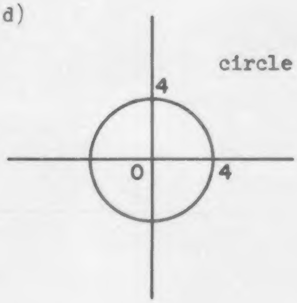
(b)



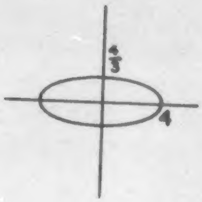
(c)



(d)

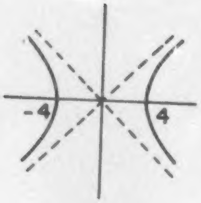


(e)



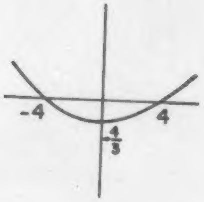
ellipse

(f)



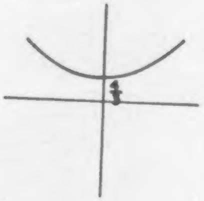
hyperbola

(g)



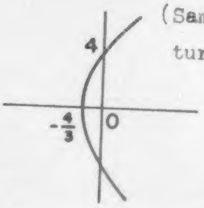
parabola

(h)



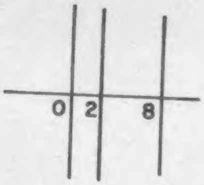
parabola

(i)



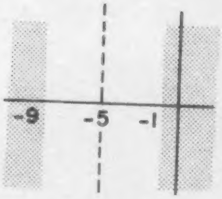
(Same as (g) turned 90°)

(j)



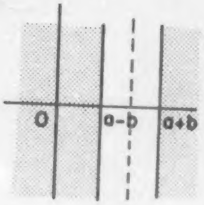
Pair of vertical lines

(k)



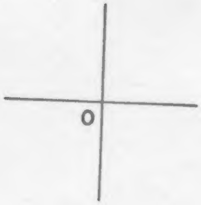
The region between but not including the vertical lines.

(ℓ)



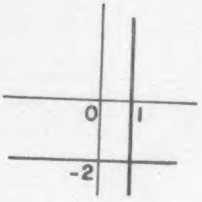
The entire plane except.

(m)



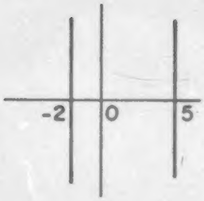
The x- and y-axes.

(n)



The two lines indicated $x = 1$, and $y = -2$.

(o)

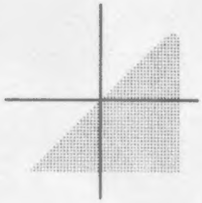


$x^2 - 3x - 10 = 0$

$(x - 5)(x + 2) = 0$

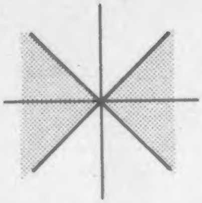
The pair of vertical lines.

(p)

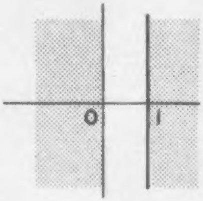


The region below the line $y = x$.

(q)



The shaded region between the lines $y = \pm x$, as shown .



$x^2 < x^2$ is equivalent to $x^2 - x > 0$, or $x(x - 1) > 0$.

This inequality is true for all x except for $0 \leq x \leq 1$. The graph is the entire plane except region between the vertical lines.

4. We do not supply full answers here, but only enough in sketch or comment to make contact with familiar material.

   (a) Circle with radius 3 and center at the pole.

   (b) The interior of the circle in (a) above.

   (c) Since there is no negative restriction on r , the set is the entire plane. If $0 < r < 3$ the set would be the same as (b) above.

   (d) The plane outside the circle of (a) above.

   (e) The line through the pole making with the polar axis an angle of measure 2 .

   (f) Since there is no negative restriction on θ the set is the entire plane.

   (g) If $r > 0$ the graph is a spiral similar to that of Figure 5-5 but opening more rapidly. It contains the pole and crosses the polar axis to the right at 4π , 8π , 12π , ... , and to the left at (abscissas) -2π , -6π , -10π , ... If $r < 0$ the graph is the symmetric image with respect to the pole of the path just described, thus the entire graph is a double spiral opening counterclockwise and crossing the polar axis at (abscissas) 0 , 2π , -2π , 4π , -4π, 6π , -6π , ...

(h) The entire plane. Compare the polar and rectangular conditions: $x = y$ gives a line, and $x < y$ a half-plane; $r = \theta$ a spiral, and $r < \theta$ the whole plane.

(i) Two lines through the origin, $\theta = 2.1$ and $\theta = 1.9$.

(j) The annular region between two concentric circles of radii 4.9 and 5.1 with centers at the pole.

In the next few solutions we supply a familiar equivalent equation in rectangular coordinates related in the obvious way, to polar coordinates. The graphs for parts (k) ... (q) are all lines, and in each case the absolute value of the numerator is the distance from the pole to the line.

(k) The line $y = 6$.

(ℓ) The line $x = -3$.

(m) The line $x = -2$.

(n) The line $x = 5$.

(o) The line through $(\sqrt{2},0)$ with slope 1 .

(p) The line through $(-4\sqrt{2},0)$ with slope 1 .

(q) We take $0 \leq b \leq 2\pi$. If $b = 0$ the graph is the line $y = a$; if $b = 2\pi$ the graph is the line $y = -a$. If $b = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ the graph is the line $x = -a$ or $x = a$, respectively. If b has any other value in the indicated domain the graph is the line through $(-a \csc b , 0)$, with the slope $\tan b$.

(r) Polar inequalities must be carefully analyzed. In this case if $0 < \theta < \pi$ the graph is the region above the line $y = 1$. If $\theta = \pi$ there is no value of r for which $r > \frac{1}{\sin \theta}$ since $\frac{1}{\sin \theta}$ is not defined then. If $\pi < \theta < 2\pi$ then the graph contains every point which is below the line $y = 1$ and on any line which intersects the line $y = 1$ and which goes through the origin. That is, this part of the graph is the region below the line $y = 1$, excluding the two half-lines along the x-axis: $y = 0$, $x > 0$, and $y = 0$, $x < 0$. To summarize, the graph of $r > \frac{1}{\sin \theta}$ is the entire plane except the points of the line $y = 1$ and the points of the two half-lines along the x-axis: $y = 0$, $x > 0$, and $y = 0$, $x < 0$. It is instructive to investigate, but we will not, the relation between $r < \frac{1}{\sin \theta}$ and $r \sin \theta < 1$, noting that this second inequality is related to $y < 1$.

(s) We consider $0 \leq \theta < 2\pi$. If $\theta = 0$ the graph is that part of the x-axis to the left of $x = 2$. If $0 < \theta < \frac{\pi}{2}$ we get, for $0 < r < \frac{2}{\cos\theta}$, the vertical strip above the x-axis and between the y-axis and the line $x = 2$. For this same domain, if $r \leq 0$ we get the origin and all points in the third quadrant. If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ we get the region to the right of the line $x = 2$. Since $\frac{2}{\cos\theta}$ is not defined for $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ there is no value of r defined for these values of θ . If $\frac{3\pi}{2} < \theta < 2\pi$ and $0 < r < \frac{2}{\cos\theta}$ we get the vertical strip below the x-axis and between the y-axis and the line $x = 2$. For this same domain if $r \leq 0$ we get the origin and all points in the second quadrant. To summarize, the graph we want is the entire plane except the line $x = 2$, and the two half-lines along the y-axis; $x = 0$, $y > 0$, and $x = 0$, $y < 0$. It is instructive to investigate, though we will not, the relation between $r > \frac{2}{\cos\theta}$ and $r\cos\theta > 2$, noting that this second inequality is related to $x > 2$.

(t) The pole.

5. In the discussion of related polar equations in Section 5-2 we used the fact that the point $P = (r,\theta)$ has also the coordinates $(-r, \theta + \pi)$. Thus, if P is on the graph of $r = f(\theta)$ we must also have P on the graph of $-r = f(\theta + \pi)$. Then we obtained the equivalent equation $r = -f(\theta + \pi)$, but this step cannot be carried through so easily with inequalities. If the point (r,θ) is on the graph of $r > f(\theta)$, then that same point, now indicated by $(-r,\theta + \pi)$, is on the graph of $-r > f(\theta + \pi)$, but this last inequality is equivalent to $r < -f(\theta + \pi)$, and this is the related polar inequality of $r > f(\theta)$. However, the original inequality can frequently be written in the form $g(r,\theta) > 0$ for which the related polar inequality is $g(-r, \theta + \pi) > 0$ and is usually easier to handle.

(a) $r^2 = 9$

(b) $r^2 < 9$

(c) $r > -3$

(d) $r < -3$

(e) $\theta = 2 - \pi$

(f) $\theta < -\frac{\pi}{2}$

(g) $r = -2(\theta + \pi)$

(h) $r > -(\theta + \pi)$

(i) $|\theta + \pi - 2| = .1$

(j) $|-r - 5| < .1$, or $|r + 5| < .1$

(k) $r = \frac{6}{\sin \theta}$

(ℓ) $r = \frac{-3}{\cos \theta}$

(m) $r = \frac{-2}{\cos \theta}$

(n) $r = \frac{5}{\cos \theta}$

(o) $r = \frac{1}{\cos(\theta + \frac{\pi}{4})}$

(q) $r = \frac{a}{\sin(\theta - b)}$

(r) $r < \frac{1}{\sin \theta}$

(s) $r > \frac{2}{\cos \theta}$

(t) $r = 0$

6. (a) $y = x^2 - 2x + 2$

(b) $x - 2y + 4 = 0$

(c) $2y = x + xy$

(d) $x^3 = y^2 + xy$

(e) $y = x^2 - 2$

(f) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(g) $\frac{(x - 2)^2}{9} + \frac{(y - 4)^2}{25} = 1$

(h) $4y^2 = x^2(4 - x^2)$

(i) $\frac{1}{x^2} + \frac{1}{y^2} = 1$

(j) $x^2 = 16y^2(1 - y^2)(1 - 2y^2)^2$

7. $x = 3 - \frac{3}{5}t$,

$y = 7 - \frac{4}{5}t$.

8. $x = 84t$,
$y = 288t$.

9. When $t = 3$, $A = (8,0)$, $B = (-1,14)$, $d(A,B) = \sqrt{277}$.
When $t = 5$, $A = (14,-2)$, $B = (-5,16)$, $d(A,B) = \sqrt{685}$.

10. When $t = 2$ $P_1 = (x_1 + 2\ell_1, y_1 + 2m_1)$, $P_2 = (x_2 + 2\ell_2, y_2 + 2m_2)$,

$$d(P_1P_2) = \sqrt{(x_1 - x_2 + 2\ell_1 - 2\ell_2)^2 + (y_1 - y_2 + 2m_1 - 2m_2)^2} \ .$$

11. (a) $x = \cos(\frac{\pi}{2} + 6\pi t)$, $y = \sin(\frac{\pi}{2} + 6\pi t)$.

(b) $x = \cos(-\frac{\pi}{2} - 4\pi t)$, $y = \sin(-\frac{\pi}{2} - 4\pi t)$.

(c) $x = \cos(-\frac{\pi}{6} + 2\pi t)$, $y = \sin(-\frac{\pi}{6} + 2\pi t)$.

(d) $x = \cos(\pi - 8\pi t)$, $y = \sin(\pi - 8\pi t)$.

(e) $x = \cos(\frac{7\pi}{6} + \pi t)$, $y = \sin(\frac{7\pi}{6} + \pi t)$.

12. We give the time in seconds and the angular position in terms of θ only. The rectangular coordinates of the position are $(\cos\theta, \sin\theta)$.

(a) $\frac{1}{10}$, $(\frac{11\pi}{10})$

(b) $\frac{2}{3}$, $(\frac{\pi}{2})$

(c) $\frac{1}{28}$, $(\frac{5\pi}{7})$

(d) $\frac{2}{15}$, $(\frac{13\pi}{10})$

(e) $\frac{5}{18}$, $(\frac{7\pi}{18})$

(f) $\frac{3}{8}$, (0)

(g) $\frac{1}{15}$, $(\frac{37\pi}{30})$

(h) $\frac{7}{60}$, $(\frac{\pi}{15})$

(i) $\frac{4}{3}$, $(\frac{\pi}{2})$

(j) $\frac{11}{54}$, $(\frac{37\pi}{27})$

13. Assume that it starts from its farthest right position

$$\begin{cases} x = 4 + 3\cos 4\pi t \ , \\ y = 5 + 3\sin 4\pi t \ . \end{cases}$$

If, when $t = 0$ it starts from the angular position θ relative to its center, then the equations of motion are

$$\begin{cases} x = 4 + 3\cos(4\pi t + \theta) \ , \\ y = 5 + 3\sin(4\pi t + \theta) \ . \end{cases}$$

14. Assume it starts from the angular position θ relative to its center. Then

$$\begin{cases} x = -1 + 2\cos(\theta - 2\pi t) \ , \\ y = \qquad\quad \sin(\theta - 2\pi t) \ . \end{cases}$$

191

15. These are all circular paths with center at the center of the clock. We give the radius, angular position of starting point, direction of rotation, and angular velocity in revolutions per minute.

(a) 4 , 0 , counterclockwise, 2 rpm.

(b) 6 , $\frac{\pi}{2}$, counterclockwise, 3 rpm.

(c) 10 , π , clockwise, 5 rpm.

(d) 8 , π , counterclockwise, 2 rpm.

(e) The given equations are equivalent to

$$\begin{cases} x = 2 \cos \left(\frac{\pi}{2} - 2\pi t\right) , \\ y = 2 \sin \left(\frac{\pi}{2} - 2\pi t\right) ; \end{cases}$$

therefore the motion is as above: 2 , $\frac{\pi}{2}$, clockwise, 1 rpm.

16. (a) $\begin{cases} x = 5 \cos \theta , \\ y = 3 \sin \theta . \end{cases}$

(b) $\begin{cases} x = 3 \cos \theta , \\ y = 4 \sin \theta . \end{cases}$

(c) $\begin{cases} x = \sqrt{6} \cos \theta , \\ y = \sqrt{5} \sin \theta . \end{cases}$

17. (a) The path of P is a cycloid with parametric equations

$$\begin{cases} x = a(\theta - \sin \theta) , \\ y = a(1 - \cos \theta) . \end{cases}$$

We assume the following: $a = 12$ inches; the wheel rolls from left to right; x is measured in inches along the road to the right from the first contact point of P ; y is measured in inches above the road; θ is the angle of rotation measured clockwise from the 6 o'clock position to the position of P ; $\theta = \omega t$ where t is measured in seconds and $\omega = 3$ rps $= 6\pi$ radius per second. Our equations are:

$$\begin{cases} x = 12(6\pi t - \sin 6\pi t) , \\ y = 12(1 - \cos 6\pi t) . \end{cases}$$

(b) The path of Q is a curtate cycloid whose equations were derived in the solution to Challenge Exercise 2 on page 18.
The equations of the path of Q are

$$\begin{cases} x = 12(6\pi t) - 6 \sin (6\pi t) , \\ y = \quad 12 \quad - 6 \cos (6\pi t) . \end{cases}$$

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